No. IX.

On the Motion of Solids on Surfaces, in the two Hypotheses of perfect Sliding and perfect Rolling, with a particular Examination of their small Oscillatory Motions. By HENRY JAMES ANDERSON, M.D. Professor of Mathematics and Astronomy, in Columbia College, New York.—*Dated 10th Nov.* 1827. *Laid before the Society 4th Jan.* 1828.

I.

THERE are few branches of Mechanical Philosophy as interesting in every point of view as the theory of Oscillatory Motion. From the minutest vibrations of a harp-string to the magnificent oscillations of a planet's axis, there are an infinite number of analogous phenomena remarkable for their curious properties or important uses. The common pendulum, that little instrument which has rendered such essential service to science and the arts, and will soon, in the hands of the skilful observer, unfold to us the internal constitution of our globe, and give a clue to the process by which it has acquired its present state, is itself indebted for its accuracy to the incessant superintendence of a watchful mathematical analysis. The science of Acoustics in all its parts, the varied phenomena of the tides, the theory of Saturn's ring, that wonder of the solar system, and the philosophical explanation of the stability and harmony of the celestial motions, are in fact

but different applications of this extensive branch of Demonstrative Mechanics. What adds to the interest and value of this subject is the circumstance that a large class of oscillatory motions, namely those of any rigid system whatever whose points depart but little from the position which they occupy when at rest, has been found susceptible of complete determination, by means of which the position of the bodies composing the system, may be expressed (to use the language of analysis) in finite functions of the time. The general problem is one, however, of the greatest difficulty, and even approximate solutions can rarely be obtained except when the conditions of the question restrict within near limits some of the variations of the system. Every contribution, therefore, however trifling, to this branch of analysis, is entitled to a favourable reception, and it is this reflection which encourages me to offer to the Society the fruits of an attentive consideration of some portions of this subject. The memoir which I have ventured to present to them is a general dissertation upon the Dynamics of solids on supporting surfaces, in the two hypotheses of perfect sliding and perfect rolling, with a special consideration of the laws of their oscillatory motions. The formulæ which I have given, besides their use in a variety of geometrical and mechanical speculations, conduct as it will be found to a complete solution of the problem of the oscillations of a supported body of any form and law of density whatever revolving on a plane or spherical surface with any initial velocity compatible with small deviations of the natural vertical of the body from its position when at rest; supposing either the absence of all friction or the action of a friction which prevents all sliding motion, but which allows the body, at the same time that it revolves round the normal, to roll in all directions from the variable point of contact. The same formulæ will conduct to the solution of a great variety of analogous problems, in which the excursions of some part of the system are confined to the immediate neighbourhood of its equilibrium position. They are susceptible moreover of easy adaptation to any hypothesis of

friction, and may readily be extended to the cases in which there are several supporting surfaces, even when these surfaces are themselves in motion.

It was my original intention to prefix to the following dissertation a detailed history of the problem of the motion of a rigid body, with an account of the successive advances which have been made from the time of Galileo to the present day towards a complete determination of the phenomena of oscillating systems. The scantiness of the New York libraries with respect to scientific works, and the impossibility under which my engagements lay me of personally consulting the more copious collections of Boston and Philadelphia, to say nothing of the fact that some of the materials of such a task are not to be found in America, and only on rare occasions to be procured from Europe, have compelled me to defer until a better opportunity the execution of this part of my first plan. I shall therefore content myself at present with a very brief preliminary retrospect of what has been already done in connexion with the subject of the following communication.

Galileo appears to have been the first who considered in a mathematical point of view even the simplest cases of the problem before us, the descent of a material point along a straight line inclined to the horizon[1], and its oscillations in the arc of a vertical circumference[2]. In the first of these two cases he succeeded in defining the motion of the point; in the second, he was far from attaining the same result, and in both the resistances of friction and the air were carefully excluded. The well known law of oscillation round a horizontal axis of support, first conjectured rather than demonstrated by Descartes in the cases of plane surfaces vibrating *in latus*[3], and afterwards generalized by the celebrated Huyg-

[1] De motu naturaliter accelerato. Opere di Galileo Galilei. Milano, 1811. Vol. viii. p. 266—306. The first edition of the *Dialogues* of Galileo is that of Leyden, 1638.

[2] Opere di Galileo. Vol. viii. p. 153—160.

[3] Renati Descartes Epistolæ. Amstelodami, 1683. Epistola LXXVII. Ad Mersennum. March 2, 1646.

hens[4], was finally in 1703 deduced by James Bernoulli, from principles which have never been contested[5]. In the mean time, Newton, in his Principia, had begun to calculate in certain cases the effects of resistance in retarding the motion of points along cycloidal arcs, had reduced to the method of quadratures, the determination of their motion along curves whose planes pass through the centre of force, and had furnished general principles which served afterwards to facilitate the solution of the problem of the motion of a heavy point on surfaces of revolution[6]. In the same work too, Newton had investigated the duration of the pulses of air and the undulations of water, and had laid the foundations of the true theory of the tides[7]. Leibnitz and the elder Bernoullis had also discussed with success several interesting cases of the descent of a material point along given or required curves[8], but no mathematician appears to have had regard to the form and rotation of the supported mass, until John Bernoulli, late in life, proposed the problem of what he called the *oscillations of titubating bodies*[9]. In this problem none but the very small oscillations are considered, and the body is supposed to rock without sliding about an invariable axis, the surface of support being either a plane or the concave or convex side of a horizontal cylinder. After investigating the general formula, Bernoulli calculates the case in which the rocking body is the segment of a sphere or parabolic conoid. This rolling

[4] Hugenii Horologium Oscillatorium. Parisiis, 1673. Pars Tertia, Prop. V.

[5] Démonstration génerale du Centre de Balancement ou d'Oscillation tirée de la nature du Levier. 15 Mars, 1703.—Histoire de l'Académie Royale des Sciences. Année MDCCIII. Paris, 1720, p. 78.—Jacobi Bernoulli opera. Genevæ, 1744. Vol. ii. p. 930.

[6] Principia, Lib. II. Prop. XXV.—XXXI.—Lib. I. Prop. LIV.—Lib. I. Prop. LV.

[7] Principia, Lib. II. Prop. L.—Prop. XLVI.—Lib. III. Prop. XXIV.—Prop. XXXVI. XXXVII.

[8] Acta Erud. Lips. 1694, p. 276. 364. 394. Jac. Bern. Op. p. 601. 627.—Leibnitii et Bern. Com. Epis. Vol. i. 23. 34. 167. 286.—Joh. Bern. Op. Vol. i. 120. iii. 486.

[9] De Oscillationibus Corporum titubantium super superficie aliquâ immobili. Joh. Bern. Op. Lausannæ et Genevæ, 1742. Vol. iv. p. 296. This paper was written posterior to the year 1738. John Bernoulli was at that time 72 years of age.

without sliding (the *pura provolutio* of Leibnitz[10]) will result, it is true, for small motions from the usual hypotheses of friction, but without some condition of this kind the body would slip or slide as well as rock. Euler is the first who made this remark in the seventh volume of the Commentaries of St Petersburg (1740), where he gives an improved solution[11] of Bernoulli's problem, but does not appear to have been able, at that time, to determine what would take place if the body were left free to slide as well as to roll. Euler acknowledged his embarrassment to D'Alembert in a letter to him dated 1746, and it is to the latter mathematician that we owe the first successful investigation of the problem when the surfaces in contact are polished to a perfect smoothness. This solution is given by D'Alembert in the second edition of his *Traité de Dynamique*, published in 1758, and is offered by him as an instance of the utility of his now celebrated principle[12]. His method is then applied to the case in which the horizontal plane opposes, by its roughness, a given degree of resistance to the sliding motion, but the oscillations are still only of the kind in which the axis of rotation retains throughout the motion its original direction. This is a condition, however, which restricts the problem to a case comparatively simple, for it is manifest that in general the axis of rotation will change continually its position in space, and the body must be considered as subject, not only to roll from side to side, but also to pitch backward and forward, and at the same time to whirl around the perpendicular drawn to the surface at the point of contact. But before the triple rotation of a supported body could be determined, it was necessary to investigate the phenomena of the rotation of a free body, to which constrained

[10] G. G. L. De lineæ super lineâ incessu, ejusque tribus speciebus, motu radente, motu provolutionis, et composito ex ambobus. Jan. 1706. Act. Erud. Lips. 1706, p. 10.

[11] De minimis oscillationibus corporum tam rigidorum quam flexibilium, methoda nova ac facilis. Com. Acad. Petrop. 1740, p. 108.

[12] Des Corps qui vacillent sur des plans. Traité de Dynamique, 1796, p. 186.

rotation can always be reduced by regarding as accelerating forces the unknown reaction of the point or surface of support. Newton, whose name it is necessary to mention in the history of almost every interesting or important speculation in Mechanical Philosophy, is the first who attempted to deduce from mathematical principles the laws of these peculiar motions as they exhibit themselves in that most remarkable exemplification of them, the Precession of the Equinoxes[13]. The singular sagacity of this extraordinary man seems to have protected him from an erroneous result, amidst a number of precarious and sometimes inaccurate assumptions to which the tediousness and barrenness of the geometric method probably forced him to resort. An amended solution of this problem was given by D'Alembert in 1749, with all the developments and verifications which the possession of a powerful analysis had brought within his reach[14]. The treatise in which this subject is discussed contains at the same time every thing that is necessary for reducing in all other cases the general problem of the free motion of a rigid body to its six differential equations. This reduction was in fact accomplished by the same author in a memoir which he announced in 1758 as prepared for the press, but which was not actually published until 1761, in the first volume of his *Opuscules Mathématiques*[15]. The results here obtained, and to a certain extent the manner of obtaining them, differ from the methods and formulas of more recent authors in little else than the improved selection and arrangement of the symbols now employed. In this respect D'Alembert was in no degree superior to his cotemporaries, and indeed nothing is more striking than the contrast which exists between the profound and original views of this illustrious writer and the negligent and inelegant notation in which they are expressed. It is a little surprising that an author who has so often in his

13 Principia, Lib. III. Prop. XXXIX.

14 Recherches sur la Précession des Equinoxes et sur la Nutation de l'Axe de la Terre dans le Système Newtonien. Paris, 1749.

15 Du Mouvement d'un Corps de Figure quelconque, animé par des forces quelconques. Opusc. Math. Vol. i. 1761, p. 74.

philosophical writings pointed out the influence which words have upon our thoughts should have studied so little the advantages of symmetrical and well selected symbols. It seems reasonable to suppose too, that as the general speculations of mechanical science refer equally to the three dimensions of space, the formulas would naturally arrange themselves in three sets similar in their form and in the process of their derivation; an arrangement which would be favoured by the method taught long since by Daniel Bernoulli and Euler[16] of separating the motion of a body into the progression of its centre of gravity and the rotation round that centre, those two constituents of the motion being absolutely independent of each other. There were however good reasons for not adopting at that time this threefold division of algebraic symbols. The most interesting application of the calculus was the investigation of the celestial motions, and analysts therefore employed the astronomical elements of position, which have not the same reference to the three parts of space. Nevertheless the preparations for a more symmetrical analysis had been made by John Bernoulli in 1715, Euler in 1736, and Maclaurin in 1742. The first of these three authors had employed, in defining the position of the points of a curve surface, three rectangular coordinates[17], the second had adopted this method for the purpose of following the motion of a

[16] Commentar. Acad. Petropol. 1737.

[17] Leib. et Bern. Com. Epis. Tom. II. p. 345. The invention of this method is ascribed by some to Euler and by others to Maclaurin. The following extracts from John Bernoulli's letter and Leibnitz's reply, while they bar all claims in favour of the former two, make it somewhat doubtful to which of the latter the merit is to be ascribed. "Intelligo per superficiem curvam datam, cujus singula puncta determinantur (sic ut lineæ curvæ data puncta) per ordinatas tres x, y, z, quarum relatio datâ æquatione exprimeretur: sunt autem tres illæ coordinatæ, nihil aliud quam tres rectæ ex quolibet superficiei curvæ puncto perpendiculariter ductæ in tria plana positione data, et se mutuo ad angulos rectos secantia. Sit æquatio inter coordinatas, exempli gratiâ, hæc $xyz = a^3$. Feb. 6, 1715." To which Leibnitz replies, "Doctrinam de æquationibus localibus trium coordinatarum, seu de locis vere solidis, olim aggredi cœpi, eorumque intersectiones seu curvas etiam non planas; sed prosequi non vacavit. Operæ pretium faceret qui studium impenderet. Apr. 9, 1715."

point[18], and along with Maclaurin had resolved velocities and forces in the direction of these coordinates[19].

Euler had observed before Maclaurin that all forces whatever soliciting a point might be resolved in three directions parallel to three fixed rectangular coordinates. He merely employed these however for the purposes of immediately resolving the forces again into three others also rectangular but not fixed, the *tangentialis*, *normalis premens*, and the *normalis deflectens*. Maclaurin appears to have been the first who endeavoured to turn to account the advantages of having the forces fixed in their directions, but the geometrical methods to which he in common with all his countrymen were unfortunately attached, made it impossible for him to realize to any extent the benefits of this arrangement.

It became an easy matter then to reduce to a regular form the calculus of the motion of a point, but it was by no means so obvious what were the three elements which were equally concerned in defining the rotations about the centre of gravity. The formulas which were first invented for this purpose were given by Euler in 1750, and may safely be pronounced among the expressions in the science most remarkable for their simplicity and absolute generality[20]. In the perfect form in which they came at once from the hands of Euler, they have been extensively employed by later mathematicians, and particularly by Lagrange in his *Mécanique Analytique*. A year before the publication of this paper, Euler had given a solution of the problem of the compound rotation of the earth[21], which he acknowledges, in a memoir on the same subject inserted in the Transactions of the Berlin

18 Mechanica analyticè exposita. Auct. Eulero. 1736. Tom. I. p. 339. 341.

19 Mechanica, Tom. II. 477.—Treatise of Fluxions, by Colin Maclaurin. Edinburgh, 1742, p. 391, § 470.

20 Découverte d'un nouveau principe de Mécanique. Mémoires de l'Académie Royale des Sciences de Berlin. Tome VI. 1750.

21 Recherches de la Précession des équinoxes, et sur la nutation de l'axe de la terre. Mémoires de l'Acad. de Berl. Tome V. 1749.

Society[22] for 1750, had been composed after a perusal of D'Alembert's Treatise of 1749. The simplifications introduced by the discovery of the properties of the natural axes of rotation by Segner[23] in 1755 contributed materially to improve the form and manageableness of the equations of rotatory motion, and in 1758, Euler had made such advances in this theory, that the problem of the general motion of a free rigid body animated by no accelerating forces, or in other words agitated only by the inertia of its particles—a problem of some celebrity in the history of mathematics,—at last yielded to the power of the calculus and to the penetrating genius of its accomplished master[24]. Two years afterwards, Euler resumed the consideration of this subject, in an interesting paper in which he applied to a variety of curious problems the theory of the Segnerian axes[25]; and finally in 1761, John Albert Euler, in a prize dissertation on the stowage and ballasting of vessels, solved by a method evidently imitated from his father's, the problem of a rigid body not solicited by accelerating forces[26]. The analysis employed in these solutions though subtle and profound is certainly deficient in that directness and precision so difficult to attain in a new and complicated subject. To remedy this imperfection, D'Alembert, in a paper written in 1762, though not published until six years afterwards[27], derives the results of Euler from the principles laid down in the first volume of his *Opuscules*[28], by a process so remarkable for its simplicity and beauty, that Lagrange has adopted and inserted it with an improved nota-

22 Avertissement au sujet des recherches sur la précession des équinoxes. Mém. de l'Acad. de Berl. Tome VI. 1750.

23 Specimen Theoriæ Turbinum. 1755.

24 Du Mouvement de Rotation des corps solides autour d'un axe variable. Mém. Acad. Berl. Tome XIV. 1758.

25 Du mouvement de rotation d'un corps solide quelconque, lorsqu'il tourne autour d'un axe mobile. Mém. Acad. Berl. Tome XVI. 1760.

26 Histoire de l'Académie des Sciences de Paris. Prix. 1761.

27 Du mouvement d'un Corps de figure quelconque. Opuscules, Tome IV 1768. p. 32.

28 Opuscules, Tome I. 1761. p. 74—103.

tion in the second volume of his *Mécanique*[29]. It is remarkable that in considering this variety of the problem, Landen, an English mathematician of excellent abilities, found himself unable to comprehend its principles, after fourteen years of earnest and almost unremitted efforts to overcome its difficulties, and that too with the solutions of Wildbore, Frisi, Euler and D'Alembert before him. In opposition to these writers he contended to the very day of his death that a correct analysis would give a constant angular rotation about the instantaneous axis.

The latter part of D'Alembert's memoir is occupied with the general equations when any accelerating forces are proposed, and contains some valuable extensions and simplifications of the formulas he had given before. It was now Euler's turn, however, to take the lead. In 1765, he had brought the general equations of rotatory motion into the form in which they are presented by Laplace in the first volume of the *Mécanique Céleste*[30], and there is an acknowledgment in the fifth volume of the same work[31], that the equations of Euler appear to him to be the very simplest which it is possible for the science to obtain. The work in which these formulas are given[32] contains two interesting applications, having some connexion with the subject of the present essay; the determination of the motion of a heterogeneous sphere on a horizontal plane, and a similar inquiry with respect to the motion of certain bodies, a given point in which remains in contact with the plane. Of these I shall speak more particularly hereafter.

The general results of Euler are obtained by the aid of the discovery of Segner. As the motions of a system, however, flow necessarily from its state at a given time and the forces by which it is solicited, it seems fair to demand a solution of the problem in which recourse shall not be had to the pro-

29 Mécanique Analytique, Tome II. 1815. p. 261—263.
30 Méc. Cél. Tome I. p. 74.
31 Méc. Cél. Tome V. p. 255.
32 Theoria motûs corporum solidorum seu rigidorum. Rostoch. 1765.

perties of the Segnerian axes. This was first effected by Lagrange[33] in the Memoirs of the Academy of Berlin for 1773. In the course of this solution, which is repeated with an improved notation in the *Mécanique Analytique*, the well known values of the resolved angular velocities in terms of the coordinates and resolved velocities of the body's poles, are given first as mere analytical abridgments, and made afterwards to exhibit their geometrical signification; a method which this author has followed on various other occasions. Nine years before this, however, Lagrange had considered another highly interesting case of planetary oscillation, the librations of the moon. His memoir on this subject was crowned by the Academy of Sciences in 1764 and will ever be memorable in the annals of Demonstrative Mechanics as containing the application of the beautiful principle of *virtual velocities* in all its simplicity and power to the most general speculations of Dynamical Philosophy[34]. Combined with the great theorem of D'Alembert, this principle dispenses altogether with the slow and enforced aids of Geometry, and leads the analyst at once from the definition of velocity and force safely and rapidly to the most recondite secrets and the most elevated regions of the Science. In the Berlin Memoirs for 1780, Lagrange resumed the whole subject, and in an admirable dissertation regarded by himself as the most finished of his productions, he terminates in formulas which delineate, in all their intricate variety, the motions of our satellite, for ages without number past and to come. These expressions are the results of a skilful transformation of the general equations in the case of rotation round a body-axis which forms with its mean direction a very small but variable angle, taking into account the figure which the moon

[33] Nouvelle solution du Problème du Mouvement de Rotation d'un Corps. Nouv. Mém. Berl. 1773.

[34] Recherches sur la libration de la Lune. Hist. Acad. Par. Prix. Tome IX. 1764.

must have acquired in the highly probable hypothesis of its original fluidity[35].

After the problem of free rotation had been solved, nearer approaches were made to the determination of the motion of a supported body. D'Alembert, who had briefly given in the first volume of his *Opuscules* the modifications of his general formulas applicable to this case, resumed the inquiry in the fifth volume of the same work[36]. For this purpose he undertakes a general solution of the question already considered by Euler. A body is supposed to be sustained by one of its points upon a plane, and the circumstances of the motion are required. The resulting differential equations are, however, so involved, that the author evidently abandons in despair all idea of obtaining the necessary integrations. A variety of simplifications and restrictions are then introduced with a view to obtain cases admitting of first integrals. The line which joins the centre of gravity and the point of support is supposed to be a principal axis, and the point is supposed to move without friction on a horizontal plane, the mode of considering the resistances of friction and the inclination of the plane being nevertheless laid down though found to lead to unmanageable results. On the whole, D'Alembert is far from having solved any but the simplest cases of this problem, though he appears to have proceeded somewhat farther than any of his cotemporaries.

Euler, who had in the earlier volumes of the Commentaries of the St Petersburg Academy considered, in conjunction with Daniel Bernoulli, the effects of friction in retarding the motion of polyhedral solids and homogeneous cylinders on inclined planes[37], turned his attention a few years before his death to some varieties of the general problem of greater difficulty than these. His first memoir on this subject is divi-

35 Théorie de la libration de la Lune. Nouv. Mém. Berl. 1780.

36 Sur le mouvement des Corps qui tournent. Opusc. Tome V. 1768. p. 489.

37 De descensu corporum super plano inclinato.—De motu corporum super plano horizontali aspero. Com. Acad. Petrop. Tom. XIII. 1751.—De frictione corporum rotantium. Novi Com. Acad. Petr. Tom. VI. 1761.

ded into two dissertations; treating of the oscillations of a heterogeneous vertical circle rolling first without and then with friction upon another vertical circle of support[38]. The entire paper is a favourable specimen of the characteristic perspicuity of Euler, and contains the solution of the problem of the small pendular motions of the body, comprised in two equations expressing in finite terms, the coexisting oscillations of the centre of gravity around the centre of the rolling circle, and of this centre around the centre of the circle of support. The integrations are effected by an application of rules which Euler had himself laid down forty years before[39] in discussing the coexisting oscillations of a jointed pendulum or string of weights, a problem of which John Bernoulli had previously proposed and resolved the simplest case, namely, that in which all the weights cross the vertical at the same instant of time[40]. Euler's solution of the general problem of the jointed pendulum stands precisely in the same relation to Bernoulli's that D'Alembert's essay on the vibrations of a tense string does to the original paper of Brook Taylor, and must be regarded as constituting an era not only in mechanical but equally so in analytical science. The singular laws of coexisting oscillations which Daniel Bernoulli had already

[38] De motu penduli circa axem cylindricum fulcro datæ figuræ incumbentem mobilis, remota frictione. Dissertatio prior. Acta Acad. Petrop. 1780, p. 133. De motu penduli, &c. habitâ frictionis ratione. Dissertatio altera, p. 164. This subject is continued in one of the numerous posthumous memoirs of Euler. Nova Acta, Tom. VI. 1778. The friction is here supposed to prevent all sliding. A general investigation requires the consideration of a friction proportioned to the pressure. This is the basis of a dissertation of Euler's (inserted in the Nova Acta for 1783, the year in which he died),—De motu globi heterogenii super plano horizontali, ejusque motu a frictione impedito. In this paper the axis of rotation is parallel to the horizon and invariable in direction. For a more recent investigation by Poisson of this motion in the case of a homogeneous sphere rolling forward and partly sliding on a horizontal plane, see Bulletin des Sciences Math. Tome VI. 1826, p. 161. This paper proceeds on the same principles as those which form the groundwork of Euler's Essay—De effectu frictionis in motu volutorio. Acta Petrop. 1781. p. 131—176.

[39] De oscillationibus fili flexilis quotcunque ponduscülis onusti. Com. Acad. Petr. 1741.

[40] De pendulo luxato, et de ejus reductione ad pendulum simplex isochronum. Joh. Bernoulli Opera, Tom. IV. p. 302.

pointed out without being able to demonstrate, are rigorously deduced from the linear differential equations in which they are comprised; and the beautiful theory of these equations, including their complete integration in a finite series of the multiples of sines of arcs proportional to the time, is developed and explained with admirable skill. An easy application of the principles of this theory solves the problem of the oscillation of a heterogeneous circle within a circle, without friction, or what is essentially the same question, of any solid upon any suitable surface, the plane of motion being invariable; as for instance a spherical segment in a spherical cup, supposing no whirling to take place, or a pendulum with cylindrical pivots working in cylindrical collars, which is the form in which the problem is proposed by Euler himself. When the friction prevents all sliding, the oscillation is single, and is determined without reference to the theory just mentioned. The effect which this friction has in diminishing the time of a pendulum's vibrations, (along with a variety of other circumstances necessary to take into the account when the *appareil* of Borda is employed) has been also calculated by Laplace in a paper on the seconds' pendulum inserted in the *Connaissance des Tems* for 1820. His memoir is remarkable for the subtlety of the analysis, rendered necessary by the multitude of the considerations included in his calculus, but when he mentions the effect of friction without sliding as a singular and interesting result to which he had arrived, he is evidently not aware of the formulas of Euler and John Bernoulli, from either of which the same inference may readily be drawn.

In the *Acta Petropolitana* for 1782, one year before his death, Euler resumes the investigation of the problem he had considered in his *Theoria motûs corporum rigidorum.* This problem, which consisted, as I have already mentioned, in determining the motion of a heterogeneous sphere along a horizontal plane, is called by Euler himself, *quæstio maximè ardua*, and is regarded by him as inaccessible by the methods then in use, except in the case in which the centres of gravity

and of figure are supposed to coincide. This simplification is accordingly introduced, and, under the hypothesis of a friction proportional to the constant pressure, he finally obtains, after a long and complicated process, a solution of the problem, as far as the progressive motion and the velocity about the instantaneous axis are concerned, but the determination of the position of this axis in terms of the time is abandoned as absolutely unattainable[41].

The whole theory of simultaneous linear equations, so important in a large class of mechanical inquiries, was left by Euler in a formed, but by no means in a finished state. D'Alembert, in whose capacious and prolific intellect almost every branch of mathematical and mechanical philosophy seems to have found place and to have borne abundant fruit, invented, for the solution of these equations, the method of indeterminate coefficients, a method remarkable for the facility of its application, and the fertile variety of its results[42]. This method is not confined as Euler's is, to the case of constant coefficients, but brings to their least difficulties many classes of equations which previously had been considered as intractable. It was however not applied by D'Alembert to the case of variable coefficients, until Lagrange and Laplace had considered the same subject in new and interesting lights. In the memoirs of the Academy of Paris for 1772, Laplace gives with numerous developments Lagrange's process for integrating any number of simultaneous linear equations of the first order with constant coefficients, and for determining the value of the arbitrary constants, which is by no means the least difficult part of the problem. Both the memoir of Lagrange which discusses the variations of the nodes of the

[41] De motu globi circa axem obliquum quemcunque gyrantis et super plano horizontali incedentis. Acta Petrop. 1782. P. ii. p. 107.

[42] Sur l'integration de quelques équations differentielles. Opuscules, Tome VII. 1780, p. 377. D'Alembert had employed the method of indeterminate multipliers in the case of constant coefficients, thirty-two years before in the Berlin Memoirs for 1748. "La belle methode de d'Alembert (these are the words of Laplace) est sûrement une des plus ingénieuses, et des plus fécondes de l'analyse." Miscellanea Taurinensia, Tom. IV. 1766, p. 273.

planets and of the inclinations of their orbits[43], and that of Laplace which is extended so as to include all their variations, whether periodical or secular[44], are alike remarkable for the analytical treasures they contain and the singular success with which this purely intellectual apparatus is made to declare the minutest and most prolonged of the celestial oscillations.

In 1788 Lagrange published his *Analytical Mechanics*. The first paragraph of the fifth section of the first edition of this work is a masterly investigation of the small oscillatory motions of any system of bodies round the places of their rest. The great generality of this solution, along with its useful applications and manageable formulas, render it altogether one of the most important contributions ever made by mathematics to mechanical philosophy[45]. The *equilibrium* positions of the elements are supposed, in Lagrange's dissertation, to be determinate and unique; that is, the system is supposed such that it cannot change its position without departing from a state of equilibrium. It is manifest however that in a large variety of cases, a system of material points may have a *range*, more or less extensive, in any part of which it will remain at rest. If the analysis of Lagrange had been made to comprehend, as far as that is practicable, the motions of a system in the immediate neighbourhood of its *range of equilibrium*, the subject would have been exhausted, and the limits of the science in no small degree enlarged.

After Huyghens and James Bernoulli had completed the

43 Recherches sur les équations séculaires des mouvemens des nœuds et des inclinaisons des Orbites des Planètes. Mém. Acad. Paris, 1774, p. 117. This paper, though of posterior date, is quoted by Laplace in the memoir following :—

44 Recherches sur le calcul intégral et sur le système du monde. Mém. Acad. Paris, 1772. P. ii. p. 293.

45 It may be well to mention for the benefit of those who may find it useful to employ these formulas, that by some oversight on the part of Lagrange the values of all the bracketted coefficients in the final differential equations are deficient in all the quantities which arise from having regard to the terms of the second order in the developments of the coordinates of the elements. In the American Journal of Science and Arts for July—Sept. 1826, p. 398, I have given the terms necessary to complete the values of these coefficients, with some remarks as to the best form of the function which expresses the finite action of the impressed forces on any one of the corpuscles of the system.

theory of oscillations round a constant axis, Clairaut in 1735 generalized the doctrine of the simple pendulum, in an able investigation of its conical vibrations, in which the effects of an oblique impulse were for the first time subjected to mathematical determination[46]. The results for the cases in which the weight describes a circle either vertical or horizontal were deduced as corollaries from the general formulas, and shown to be coincident with the conclusions to which Huyghens had already arrived for these simpler cases of the question. A more difficult problem still remained. When a pendulous body hangs by a fixed point about which it may turn freely in all directions, its motion will be affected not only by the obliquity of the impulse by which it is set in motion, but also by the rotation of the pendulum around the line which joins the sustaining point and the centre of gravity, so that even when this axis is dropped vertically from a state of rest with the body revolving around it, this rotation will be sufficient, at every instant of the motion, to wrench (as it were) the axis from the direction in which it would move if it were left at the same instant to vibrate by itself. Up to the present time no solution of this problem has been given for finite oscillations, and even for oscillations infinitely small, none was given until Lagrange published, in the first edition of his *Mécanique Analytique*, an ample dissertation on the subject. After a general investigation of the free rotation of a rigid body, in which the author skilfully combines all the advantages of the various methods he had previously invented, he proceeds to the examination of the well known case in which the body *pirouettes* by virtue of the inertia of the elements alone. After a masterly detail of all the circumstances of this case, Lagrange enters upon the discussion of the general motions of a heavy body *pirouetting* about a fixed point not the centre of gravity, and advances as far towards

[46] Examen des differentes Oscillations qu'un corps suspendu par un fil, peut faire lorsqu'on lui donne une impulsion quelconque. Mém. Acad. Par. 1735, p. 281.

a solution as it is possible to proceed in the present state of the Calculus. The case however in which the natural vertical of the body makes infinitely small conical oscillations around its resting place, while the body itself revolves about this axis with any velocity compatible with such oscillations, is completely solved by means of an analysis remarkable for its brilliancy, generality and rigour. The problem, it is shown, naturally divides itself into two distinct portions, one in which the form and density of the body is absolutely arbitrary, but the rotation round the vertical small and consequently variable; the other in which the rotation round the vertical is arbitrary and consequently constant, but the form and density of the body such that the conditions requisite to constitute the natural vertical a natural axis of rotation shall be nearly, though it is not necessary that they should be exactly, fulfilled.

Poisson published his excellent *Traité de Mécanique* in 1811. In the second volume of this work, the author applies his calculus to a determination of the motions of a homogeneous ellipsoid upon an inclined plane, both surfaces being supposed perfectly smooth. The investigation does not bring the formulas within the reach of the method of quadratures, and therefore the problem cannot as yet be considered as solved[47]. The author then proceeds to give an improved solution of the question considered long before by Euler and D'Alembert, of the motion of a solid body when it is sustained upon a plane by a point fixed in the body, but moving freely along the plane. In the case in which the density and figure are symmetrical about the axis joining the centre of gravity and sustaining point, the problem is reduced to the method of quadratures, and a complete solution is given in the hypothesis of small departures of the axis from some intermediate inclination to the plane. In this solution Poisson has been followed by Prony in his *Leçons de Mécanique*

[47] This reduction, it ought to have been remarked, is easily effected when the ellipsoid becomes a spheroid of revolution.

Analytique[48], Whewell in his *Dynamics*[49], and various other authors and compilers.

It is, I think, a matter of surprize, that none of the European mathematicians should have thought of ascertaining whether the method of Lagrange might not be successfully employed in determining the variable *pirouettes* or oscillations which a heavy body bounded by a given surface will make on a given plane or in general on any given surface of support. The first solutions I have been able to find of any case whatever of this interesting question are contained in the eighth number of the New York Mathematical Diary for July 1827. The problem as proposed by Mr E. Nulty, of Philadelphia, requires a determination of all the small oscillations which can be made by the segment of a sphere in contact with a horizontal plane. Euler, as we have seen, had perfectly resolved this case, in the two hypotheses of perfect sliding and perfect rolling, as long as the motion of rotation is around an axis of invariable direction. But the motion round a variable axis he had carefully excluded, expressly on the ground of its being inaccessible to the analysis of the day. One of the solutions published in the work which I have just mentioned is by Dr Adrain, at that time Professor of Mathematics in Rutger's College, New Jersey. This solution, which regards the segment as symmetrical and moving without friction, begins with a very ingenious and direct transformation of Lagrange's general formula of Dynamics into another in which three of the variations are, as usual, variations of the coordinates of the centre of rotation, and the other three, variations of the finite angles employed by Euler and Laplace; a process which, though the most direct, has not, as far as I can ascertain, been pursued or even suggested by any other author. The facility with which this problem, as long as friction is not concerned, may be subjected to the methods and formulas of Lagrange, enabled me, in a solution

48 Leçons de Mécanique Analytique, Tome II. 1815. p. 415.

49 A Treatise on Dynamics. 1823. p. 336.

subjoined to Dr Adrain's in the same number of the Diary, to dispense with the conditions of a symmetrical density or a vertical natural axis of rotation. In the hypothesis of perfect rolling (the first instance I believe in which it has been considered in reference to an axis varying *ad libitum*) the formulas I have there given lead to a complete solution of the problem[50] considered in all the generality of which it is susceptible. It still remained to apply to oscillating bodies of any form whatever what is there remarked of bodies with a spherical areola of contact, and at the same time to have regard to the figure of the surface of support. This I have attempted in the following dissertation; with what success I leave to those who are better practised than myself in speculations of this nature, to examine and decide.

Before entering upon this subject, I beg leave simply to remark that the new words or new combinations of words occasionally employed in the following paper, have not been introduced from any idle love of innovation, but from the absolute necessity of the case. The tedious circumlocutions and the incessant repetitions to which I should have been forced without the proposed abridgments, would have extended this communication far beyond its proper limits, and would not I think have added either to its interest or perspicuity. In short, I have employed these terms precisely for the same reasons that I employ the symbols of analysis, and attach no sort of value to them after they have served my purpose, but leave them to be accepted or rejected, as those who choose to pursue this subject may happen to find it most convenient.

[50] In consequence of an error in developing the variation of the living forces due to the progression of the system, a correction (to be made by substituting $e-h$ in place of e) becomes necessary in some of the expressions at the close of the paper above referred to.

II.

Mathematical Investigation of the Motion of Solids upon Surfaces, in the Two Hypotheses of Perfect Sliding and Perfect Rolling, with a Particular Examination of their Small Oscillatory Motions.

Let us now refer, as usual, the oscillating body (M) to two systems of coordinate axes, one of them, which I shall call *space axes*, fixed in space, and originating at any fixed point (O), the other called *body axes*, invariably connected with the body and originating at any given point ($O_{,}$). Let x, x', x'', $x_{,}$, $y_{,}$, $z_{,}$, denote the coordinates of any element Dm of the body referred to these two sets of axes; ξ, ξ', ξ'', $\xi_{,}$, $\eta_{,}$, $\zeta_{,}$, the coordinates of $O_{,}$, reckoned from O, parallel respectively to the space and body axes; A, B, C, p, q, r, the moments of inertia and the velocities of rotation round the body axes; F, G, H, P, Q, R, the integrals $Sy_{,}z_{,}Dm$, $Sz_{,}x_{,}Dm$, $Sx_{,}y_{,}Dm$, $\int pdt$, $\int qdt$, $\int rdt$; X, X', X'', $X_{,}$, $Y_{,}$, $Z_{,}$, the accelerative forces in the direction of the space and body axes; and finally, the symbol d denoting the differential coefficient with respect to the time t*, let dx_{1}, dy_{1}, dz_{1}, $d\xi_{1}$, $d\eta_{1}$, $d\zeta_{1}$, d^2x_{2}, d^2y_{2}, d^2z_{2}, and $d^2\xi_{2}$, $d^2\eta_{2}$, $d^2\zeta_{2}$, denote the velocities and accelerations of Dm and $O_{,}$, in the direction of the axes of the body.

As the general formula of Dynamics is, by its nature, inde-

* I have ventured upon this modification of the usual notation, at the suggestion of a valued friend, principally with a view to save room. A single symbol of the form $\frac{\mathrm{d}x}{\mathrm{d}t}$, besides occasioning more or less of trouble and delay to the printer, evidently makes every line in which it is introduced take up more than double the space which it would occupy without it. The Roman d will be reserved (as usual in these Transactions) for simple differentials.

pendent of the direction of the axes in space, it may be presented in either of these forms, (1)

$$SDm[(d^2x + X)\delta x + (d^2x' + X')\delta x' + (d^2x'' + X'')\delta x''] = 0$$

$$SDm[(d^2x_2 + X_{/})\delta x_1 + (d^2y_2 + Y_{/})\delta y_1 + (d^2z_2 + Z_{/})\delta z_1] = 0$$

where it must be carefully recollected, that in consequence of the motion of the body axes, the variations and accelerations in the latter formula, as well as the velocities dx_1, dy_1, dz_1, $d\xi_1$, $d\eta_1$, $d\zeta_1$, belong to the class of incomplete differentials.

In these equations the variations are of different values for different elements of the body, or in other words are functions of the coordinates of Dm. It is evident, however, that before this formula can be employed, these variations will in general require to be reduced to other variations *common* to all the elements, so that, in the language of the calculus, they may be passed from under the sign S. The manner of effecting this, by a general method for all constitutions of matter and for all conditions of motion, must have been a problem of no ordinary difficulty. Mathematicians however have succeeded in this transformation by several processes equally remarkable, each of them terminating in an equation of the form

$$L\delta\alpha + M\delta\beta + N\delta\gamma + L'\delta\lambda + M'\delta\mu + N'\delta\nu = 0.$$

In all these transformations, $\delta\alpha$, $\delta\beta$, $\delta\gamma$ are the *progressive* variations common to all the particles in the direction either of the body-axes or the axes in space; but with respect to the variations $\delta\lambda$, $\delta\mu$, $\delta\nu$, there exists between these methods an essential difference which deserves to be noticed. To render this distinction the clearer, it is necessary to observe that the absolute position of a body in space involves two considerations: 1st, the position in space of some fixed point $O_{/}$ of the body, which may be denominated the *station* of the body; and 2dly, that part of the position which depends only upon the direction of the body-axes, and which, for the sake of brevity, may be called the *aspect* of the body. A body therefore may

change its station while it keeps its aspect, or it may alter its aspect while it maintains its station, these two constituents of position being entirely independent of each other. It is evident, moreover, that the station of a body depends upon three arbitrary variables, the three coordinates of $O_{,}$; whereas its aspect is a function of the nine angles which the three body-axes make with the three axes in space. As the angles which a straight line makes with axes to which it is referred, are elements of very frequent use in geometrical and mechanical speculations, I shall take the liberty, for the purpose of avoiding tedious repetitions, to call them the *axe-angles* of the line, distinguishing also between the *space-axe angles* and the *body-axe angles*; thus a, a', a'', b, b', b'', c, c', c'' (which is the usual notation) will denote the cosines of the space-axe angles of the body-axes. Between these nine cosines there exist six equations of condition, so that, in ultimate analysis, the aspect of a body will, as well as its station, depend upon the values of three independent variables. The choice of these becomes therefore a matter of importance. Euler, who must be regarded as the inventor of this interesting branch of analysis, showed as early as the year 1771, in a paper published in the fifteenth volume of the *Novi Commentarii* of the Academy of St Petersburg, under the title of *Problema algebraicum ob affectiones prorsus singulares memorabile*, how these nine quantities might be expressed in terms of three independent angles, namely, the inclination of one of the moveable to one of the fixed planes, and the distances from their intersection to an axis in each plane. The author begins by considering the question analytically; and this view of it gives rise to a problem altogether similar, with respect to the determination of sixteen quantities connected by ten analogous conditions, from which he proceeds, with his characteristic habit of gradual generalization, to extend his analysis to twenty-five quantities with fifteen connecting relations, and so on. It is only the first case of the problem that can have any application to geometry, but the whole paper is deserving of attention as furnishing one of the earliest specimens of the improved methods of modern analysis. Of all the solutions of the first case of

this problem which have since been given, there is perhaps none equal to Euler's in directness and perspicuity. The methods of obtaining the resulting formulas have however been, with great advantage, occasionally modified so as to suit particular views and purposes. It is in astronomy more especially that these three elements of aspect are most employed, for which reason they are preferred by Laplace to the three indefinite integrals, the angles P, Q, R, notwithstanding the greater symmetry which arises from the use of these three angles. It is to Euler also that we are indebted for formulas which lead to this last determination, by which the cosines of the nine angles are made to depend by the medium of differential equations on the values of the integrals P, Q, R. In the sixth volume of the Berlin Transactions for the year 1750, in a memoir entitled *Découverte d'un nouveau principe de mécanique*, Euler gave the formulas, now so well known, which express the motion of every point of a system in terms of the coordinates of the point and the motion of progression and rotation common to all the points. These expressions were employed by Lagrange in obtaining the relations by which the variations of the cosines of the axe-angles were reduced to the three variations δP, δQ, δR, or the three analogous variations of the angles of rotation round the axes fixed in space. Finally, in the Memoirs of the Academy of Turin for the years 1784 and 1785, there is a curious paper by Monge, in which, having occasion to introduce these nine cosines, he takes for the independent variables the three angles $xO_{,}x_{,}$, $x'O_{,}y_{,}$, $x''O_{,}z_{,}$, and gives without demonstration the values of the other six, expressed in terms of these three. Lacroix has inserted these results, with an accompanying demonstration, in his quarto treatise on the Differential and Integral Calculus; but I am not aware that this method of determination has been employed in Analytical Mechanics.

One of the methods by which the transformation from individual to common variations has been effected is founded on the formulæ which give the variations of the cosines a, a', a'', &c. in terms of the variations of the angles of rotation round the space-axes. This method has the advantage of leading

readily to two integrations with respect to time, thus giving at once the principles of the centre of gravity, of areas and of living forces; but does not allow of integration with respect to the dimensions of the system, without which it is obvious that the phenomena of its motion cannot in general be ascertained. For this purpose either the equations of motion obtained by this transformation must be employed to produce six others which admit of this integration, as Laplace has done, or these six must be obtained directly from the general dynamical equation by the application of the formulas involving the variations of the angles of rotation round the axes of the body, a method which was first carried fully into effect by Lagrange. In his *Mécanique Analytique*, he effects this transformation, not by a direct method, but by means of his favourite subsidiary formulas, (*Vol. I. p.* 13, edit. 1811); and in doing so he is under the necessity of warning the reader that the usual interchange of the differential of the variation and the variation of the differential would not be legitimate with respect to the quantities δdP, δdQ, δdR. The difference arising from the order in which the signs are placed (a difference obviously to be ascribed to the incompleteness of both the differential and the variation of the indefinite integrals P, Q, R,) Lagrange then carefully investigates and takes into account. In a note found among his papers after his death, and inserted in an appendix at the end of the second volume of his *Mécanique*, he carries to its results, by a direct process, the last mentioned plan of transformation, and extends his analysis to all possible systems, whether solid or not, thereby having regard to the intestine or proper motions of the particles. As the method indicated in this note appears to me to conduct to the necessary results from the simplest principles, by the directest means, and with the smallest quantity of analysis compatible with a process entirely analytical, I shall devote a page or two of this paper to the purpose of obtaining, by means of this transformation, formulas preparatory to the solution of the problem I have proposed. On this subject I think it proper to premise, that as the whole notation I have adopted refers

to the three dimensions of matter with absolute similarity, the equations will necessarily form themselves into triplets perfectly symmetrical, so that when the first of each triplet is investigated, the others will be had without calculation by changing simultaneously throughout the first triplet every letter of the triplets of notation into the letter which follows it circularly in that triplet. The same observation applies equally to the accents; to allow of which in all cases, it was necessary to alter in some respects the usual notation, which however will not be much disturbed if we make the triplets of successive accents refer to the fixed axes and the triplets of successive letters to the axes of the body.

Among the quantities which I have distinguished by symbols, there exist the following well known relations.

$$
(2) \qquad \begin{aligned}
x &= \xi + a x_{,} + b y_{,} + c z_{,} \\
x' &= \xi' + a' x_{,} + b' y_{,} + c' z_{,} \\
x'' &= \xi'' + a'' x_{,} + b'' y_{,} + c'' z_{,}
\end{aligned}
$$

$$
(3) \qquad \begin{aligned}
x_{,} &= -\xi_{,} + ax + a'x' + a''x'' \\
y_{,} &= -\eta_{,} + bx + b'x' + b''x'' \\
z_{,} &= -\zeta_{,} + cx + c'x' + c''x''
\end{aligned}
$$

$$
(4) \qquad \begin{aligned}
a^2 + a'^2 + a''^2 &= 1 & \qquad a\,b + a'b' + a''b'' &= 0 \\
b^2 + b'^2 + b''^2 &= 1 & \qquad b\,c + b'c' + b''c'' &= 0 \\
c^2 + c'^2 + c''^2 &= 1 & \qquad c\,a + c'a' + c''a'' &= 0 \\
\\
a^2 + b^2 + c^2 &= 1 & \qquad a\,a' + b\,b' + c\,c' &= 0 \\
a'^2 + b'^2 + c'^2 &= 1 & \qquad a'a'' + b'b'' + c'c'' &= 0 \\
a''^2 + b''^2 + c''^2 &= 1 & \qquad a''a + b''b + c''c &= 0
\end{aligned}
$$

$$
(5) \qquad \begin{aligned}
dP &= cdb + c'db' + c''db'' \\
dQ &= adc + a'dc' + a''dc'' \\
dR &= bda + b'da' + b''da''
\end{aligned}
$$

(6)

$$
\begin{aligned}
da &= b\,dR - c\,dQ & \qquad db &= c\,dP - a\,dR & \qquad dc &= a\,dQ - b\,dP \\
da' &= b'dR - c'dQ & \qquad db' &= c'dP - a'dR & \qquad dc' &= a'dQ - b'dP \\
da'' &= b''dR - c''dQ & \qquad db'' &= c''dP - a''dR & \qquad dc'' &= a''dQ - b''dP
\end{aligned}
$$

The equations marked (5) occur for the first time in Lagrange's memoir of 1773 referred to in the historical sketch which precedes this essay, and arise not by any derivation from their mechanical meaning, but simply as analytical abridgments naturally presenting themselves in the course of his investigations, and then afterwards examined and defined. The values of these velocities dP, dQ, dR, may however be obtained from their definitions without calculation, by means of the following simple consideration,—that the velocity round *any one* of the axes is the same with the velocity of a point (distant unity from $O_{\prime}$ in a *second* axis) estimated in the direction of the *third* axis. Thus the components, in the direction of the fixed axes, of the velocity of the point (a, a', a'') in the axis of $x_{\prime}$, being da, da', da'', its velocity in the direction of the axis of $y_{\prime}$ will be $bda + b'da' + b''da''$, which is therefore equal to dR, the velocity of rotation round the axis of $z_{\prime}$. The velocities dP, dQ are then had by changing the letters.

The following corollaries from the above formulas will be useful on a variety of occasions: (7)

$$\begin{aligned}
da^2 + da'^2 + da''^2 &= dQ^2 + dR^2 \\
db^2 + db'^2 + db''^2 &= dR^2 + dP^2 \\
dc^2 + dc'^2 + dc''^2 &= dP^2 + dQ^2
\end{aligned}$$

$$\begin{aligned}
dadb + da'db' + da''db'' &= -dPdQ \\
dbdc + db'dc' + db''dc'' &= -dQdR \\
dcda + dc'da' + dc''da'' &= -dRdP
\end{aligned}$$

$$\begin{aligned}
ad^2a + a'd^2a' + a''d^2a'' &= -(dQ^2 + dR^2) \\
bd^2b + b'd^2b' + b''d^2b'' &= -(dR^2 + dP^2) \\
cd^2c + c'd^2c' + c''d^2c'' &= -(dP^2 + dQ^2)
\end{aligned}$$

$$\begin{aligned}
ad^2b + a'd^2b' + a''d^2b'' &= dPdQ - d^2R \\
bd^2c + b'd^2c' + b''d^2c'' &= dQdR - d^2P \\
cd^2a + c'd^2a' + c''d^2a'' &= dRdP - d^2Q
\end{aligned}$$

$$\begin{aligned}
ad^2c + a'd^2c' + a''d^2c'' &= dRdP + d^2Q \\
bd^2a + b'd^2a' + b''d^2a'' &= dPdQ + d^2R \\
cd^2b + c'd^2b' + c''d^2b'' &= dQdR + d^2P
\end{aligned}$$

Much use will also be made of the subjoined equations:

$$\begin{aligned} \xi &= a\,\xi_{,} + b\,\eta_{,} + c\,\zeta_{,} \\ \xi' &= a'\xi_{,} + b'\eta_{,} + c'\zeta_{,} \\ \xi'' &= a''\xi_{,} + b''\eta_{,} + c''\zeta_{,} \end{aligned} \qquad (8) \qquad \begin{aligned} \xi_{,} &= a\xi + a'\xi' + a''\xi'' \\ \eta_{,} &= b\xi + b'\xi' + b''\xi'' \\ \zeta_{,} &= c\xi + c'\xi' + c''\xi'' \end{aligned}$$

$$\begin{aligned} dx_{1} &= adx + a'dx' + a''dx'' \\ dy_{1} &= bdx + b'dx' + b''dx'' \\ dz_{1} &= cdx + c'dx' + c''dx'' \end{aligned} \qquad (9) \qquad \begin{aligned} d^2x_{2} &= ad^2x + a'd^2x' + a''d^2x'' \\ d^2y_{2} &= bd^2x + b'd^2x' + b''d^2x'' \\ d^2z_{2} &= cd^2x + c'd^2x' + c''d^2x'' \end{aligned}$$

$$\begin{aligned} d\xi_{1} &= ad\xi + a'd\xi' + a''d\xi'' \\ d\eta_{1} &= bd\xi + b'd\xi' + b''d\xi'' \\ d\zeta_{1} &= cd\xi + c'd\xi' + c''d\xi'' \end{aligned} \qquad (10) \qquad \begin{aligned} d^2\xi_{2} &= ad^2\xi + a'd^2\xi' + a''d^2\xi'' \\ d^2\eta_{2} &= bd^2\xi + b'd^2\xi' + b''d^2\xi'' \\ d^2\zeta_{2} &= cd^2\xi + c'd^2\xi' + c''d^2\xi'' \end{aligned}$$

with similar expressions for the incomplete variations δx_{1}, δy_{1}, δz_{1}, $\delta\xi_{1}$, $\delta\eta_{1}$, $\delta\zeta_{1}$. Finally we have the following relations between the accelerative forces:

$$\begin{aligned} X &= a\,X_{,} + b\,Y_{,} + c\,Z_{,} \\ X' &= a'X_{,} + b'Y_{,} + c'Z_{,} \\ X'' &= a''X_{,} + b''Y_{,} + c''Z_{,} \end{aligned} \qquad (11) \qquad \begin{aligned} X_{,} &= aX + a'X' + a''X'' \\ Y_{,} &= bX + b'X' + b''X'' \\ Z_{,} &= cX + c'X' + c''X'' \end{aligned}$$

If we substitute now, in place of the variations and accelerations of x, x', x'' in the above formulas, their values derived from equations (2), and reduce by means of (6) and (10), we shall find

(12)

$$\begin{aligned} \delta x_{1} &= \delta\xi_{1} - y_{,}\delta R + z_{,}\delta Q \\ \delta y_{1} &= \delta\eta_{1} - z_{,}\delta P + x_{,}\delta R \\ \delta z_{1} &= \delta\zeta_{1} - x_{,}\delta Q + y_{,}\delta P \end{aligned}$$

(13)

$$\begin{aligned} d^2x_{2} &= d^2\xi_{2} - x_{,}(dQ^2 + dR^2) + y_{,}(dPdQ - d^2R) + z_{,}(dRdP + d^2Q) \\ d^2y_{2} &= d^2\eta_{2} - y_{,}(dR^2 + dP^2) + z_{,}(dQdR - d^2P) + x_{,}(dPdQ + d^2R) \\ d^2z_{2} &= d^2\zeta_{2} - z_{,}(dP^2 + dQ^2) + x_{,}(dRdP - d^2Q) + y_{,}(dQdR + d^2P) \end{aligned}$$

By the substitution of these expressions in the second general formula (1), it becomes integrable with respect to S; and if we suppose the point $O_{\prime}$ to be taken in the centre of gravity of the system, we shall have, after the obvious reductions,

(14)

$$\left.\begin{array}{l}(Md^2\xi_2+SX_{\prime}Dm)\delta\xi_1+[U+S(Z_{\prime}y_{\prime}-Y_{\prime}z_{\prime})Dm]\delta P\\(Md^2\eta_2+SY_{\prime}Dm)\delta\eta_1+[V+S(X_{\prime}z_{\prime}-Z_{\prime}x_{\prime})Dm]\delta Q\\(Md^2\zeta_2+SZ_{\prime}Dm)\delta\zeta_1+[W+S(Y_{\prime}x_{\prime}-X_{\prime}y_{\prime})Dm]\delta R\end{array}\right\}=0,$$

where

(15)

$$\begin{aligned}U&=Adp-Gdr-Hdq+(C-B)qr+F(r^2-q^2)-Gpq+Hrp,\\V&=Bdq-Hdp-Fdr+(A-C)rp+G(p^2-r^2)-Hqr+Fpq,\\W&=Cdr-Fdq-Gdp+(B-A)pq+H(q^2-p^2)-Frp+Gqr;\end{aligned}$$

which are the same expressions as those which are given by Lagrange in his first volume, although obtained by a process altogether different.

These values would be greatly simplified by referring the elements Dm to the principal axes of the body; but as the axis which is vertical when a heavy body is at rest is not in general a principal axis, it will be found necessary, in investigating the phenomena of oscillatory motion, to retain the terms multiplied by F, G, H, quantities which may I think, from their giving rise to a constant displacement of the instantaneous axis of rotation, be called with some propriety the *distorsive moments of inertia.*

If the system is free, then by equating to nought the coefficients of the six variations, we shall obtain six equations determining the progressive and rotatory motion of the body, namely, (16)

$$
\begin{aligned}
Md^2\xi_2 + SX_{,}Dm &= 0\,,\\
Md^2\eta_2 + SY_{,}Dm &= 0\,,\\
Md^2\zeta_2 + SZ_{,}Dm &= 0\,;\\
U + S(Z_{,}y_{,} - Y_{,}z_{,})Dm &= 0\,,\\
V + S(X_{,}z_{,} - Z_{,}x_{,})Dm &= 0\,,\\
W + S(Y_{,}x_{,} - X_{,}y_{,})Dm &= 0\,.
\end{aligned}
$$

of which the first three may, by means of equations (10) and (11), be made to assume the following more usual form (17)

$$
\begin{aligned}
Md^2\xi + SX\,Dm &= 0\,,\\
Md^2\xi' + SX'Dm &= 0\,,\\
Md^2\xi'' + SX''Dm &= 0\,.
\end{aligned}
$$

But if the body, as in the problem I have proposed to examine, is forced to roll or slide on a given surface, the above variations are no longer independent, and we must ascertain the influence which the progressive and rotatory motions have upon each other; or to give this question the geometrical form which the nature of variations seems essentially to require, it is necessary to determine the geometrical relations which a given limitation of position will occasion among the elementary changes of those magnitudes on which the station and the aspect of the body depend. For this purpose, let $K = 0$ represent the equation of the given supporting surface referred to the axes fixed in space, and let $K_{,} = 0$ be the equation of the surface of the given oscillating body referred to its own axes. Let L, L', L'', $L_{,}$, $M_{,}$, $N_{,}$, represent the cosines of the space and body axe-angles made by the normal common to both surfaces at the point of variable contact P, for whose space and body coordinates we may employ the symbols x, x', x'', $x_{,}$, $y_{,}$, $z_{,}$, so as to make the formulas (3) applicable to these coordinates, recollecting only that $x_{,}$, $y_{,}$, $z_{,}$ are now variable quantities. Then because the normal is at right angles to the

elements δs, $\delta s_{,}$ of the curves traced on the two surfaces by the point of contact P, and that δx, $\delta x'$, $\delta x''$, $\delta x_{,}$, $\delta y_{,}$, $\delta z_{,}$, are proportional to the cosines of the space and body axe-angles of this element, we have the two equations (18)

$$\begin{aligned} L\,\delta x + L'\,\delta x' + L''\delta x'' &= 0; \\ L_{,}\delta x_{,} + M_{,}\delta y_{,} + N_{,}\delta z_{,} &= 0. \end{aligned}$$

But the variational equations of the given surfaces are

$$\frac{dK}{dx}\,\delta x + \frac{dK}{dx'}\,\delta x' + \frac{dK}{dx''}\delta x'' = 0;$$

$$\frac{dK_{,}}{dx_{,}}\,\delta x_{,} + \frac{dK_{,}}{dy_{,}}\,\delta y_{,} + \frac{dK_{,}}{dz_{,}}\,\delta z_{,} = 0.$$

which equations, to subsist simultaneously with the other two, require that we should have (19)

$$\begin{aligned} L &= k\,.\frac{dK}{dx}, & L_{,} &= k_{,}\,.\frac{dK_{,}}{dx_{,}}, \\ L' &= k\,.\frac{dK}{dx'}, & M_{,} &= k_{,}\,.\frac{dK_{,}}{dy_{,}}, \\ L'' &= k\,.\frac{dK}{dx''}. & N_{,} &= k_{,}\,.\frac{dK_{,}}{dz_{,}}. \end{aligned}$$

Whence $k\sqrt{\left(\frac{dK^2}{dx^2} + \frac{dK^2}{dx'^2} + \frac{dK^2}{dx''^2}\right)} = \sqrt{(L^2 + L'^2 + L''^2)} = 1$;

and $$k = \left(\frac{dK^2}{dx^2} + \frac{dK^2}{dx'^2} + \frac{dK^2}{dx''^2}\right)^{-1}.$$

Similarly $$k_{,} = \left(\frac{dK_{,}^2}{dx_{,}^2} + \frac{dK_{,}^2}{dy_{,}^2} + \frac{dK_{,}^2}{dz_{,}^2}\right)^{-1}.$$

These last formulas, which are well known to mathematicians, will enable us to find the values of L, L', L'', $L_{,}$, $M_{,}$, $N_{,}$, in all cases where the surfaces are known, and thereby to put their differential or variational equations in the forms above given (18); forms which will always be found remarkably well adapted to geometrical and mechanical inquiries, from the facility with which the analytical results can be translated into the language of geometry. Between these cosines there exist the following relations: (20)

$$\begin{aligned} L &= a\,L_{,} + b\,M_{,} + c\,N_{,}\,, \\ L' &= a'L_{,} + b'M_{,} + c'N_{,}\,, \\ L'' &= a''L_{,} + b''M_{,} + c''N_{,}\,; \end{aligned}$$

$$\begin{aligned} L_{,} &= a\,L + a'L' + a''L''\,, \\ M_{,} &= b\,L + b'L' + b''L''\,, \\ N_{,} &= c\,L + c'L' + c''L''\,. \end{aligned}$$

Taking now the variations of equations (2), and recollecting that $x_{,}$, $y_{,}$, $z_{,}$ are no longer constant, we obtain (21)

$$\begin{aligned} \delta x &= \delta\xi + a\,\delta x_{,} + b\,\delta y_{,} + c\,\delta z_{,} + x_{,}\delta a + y_{,}\delta b + z_{,}\delta c\,, \\ \delta x' &= \delta\xi' + a'\delta x_{,} + b'\delta y_{,} + c'\delta z_{,} + x_{,}\delta a' + y_{,}\delta b' + z_{,}\delta c'\,, \\ \delta x'' &= \delta\xi'' + a''\delta x_{,} + b''\delta y_{,} + c''\delta z_{,} + x_{,}\delta a'' + y_{,}\delta b'' + z_{,}\delta c''\,. \end{aligned}$$

Adding these equations together, after multiplying the first by L, the second by L', and the third by L'', and then reducing by means of the formulas (20), there results

$$\left.\begin{aligned} &-(L\,\delta x + L'\delta x' + L''\delta x'') + x_{,}(L\delta a + L'\delta a' + L''\delta a'') \\ &+(L\,\delta\xi + L'\delta\xi' + L''\delta\xi'') + y_{,}(L\delta b + L'\delta b' + L''\delta b'') \\ &+(L_{,}\delta x_{,} + M_{,}\delta y_{,} + N_{,}\delta z_{,}) + z_{,}(L\delta c + L'\delta c' + L''\delta c'') \end{aligned}\right\} = 0.$$

Substituting in place of the variations of the nine cosines their

values (6), reducing by means of equations (20), and observing that the differential equations of the surfaces give us

$$L\delta x + L'\delta x' + L''\delta x'' = 0, \qquad L_{,}\delta x_{,} + M_{,}\delta y_{,} + N_{,}\delta z_{,} = 0,$$

we shall find

$$\begin{aligned} 0 = L\,\delta\xi + L'\,\delta\xi' + L''\delta\xi'' &+ (N_{,}y_{,} - M_{,}z_{,})\delta P, \\ &+ (L_{,}z_{,} - N_{,}x_{,})\delta Q, \\ &+ (M_{,}x_{,} - L_{,}y_{,})\delta R, \end{aligned}$$

which, by virtue of the relations (10) and (11), may be also presented in this form, (22)

$$\begin{aligned} 0 = L_{,}\delta\xi_{1} + M_{,}\delta\eta_{1} + N_{,}\delta\zeta_{1} &+ (N_{,}y_{,} - M_{,}z_{,})\delta P, \\ &+ (L_{,}z_{,} - N_{,}x_{,})\delta Q, \\ &+ (M_{,}x_{,} - L_{,}y_{,})\delta R; \end{aligned}$$

remarkable expressions, independent of the variations of the point of contact, and containing the required relation between the variations of the station and aspect of the body made necessary by the condition of its contact with the given surface of support. These equations are in other respects independent of the manner in which the body is forced to slide, roll, or whirl upon the surface, and are therefore true under every hypothesis of friction.

It may be well to observe that these equations, the last for example, may be obtained by another method which introduces formulas that may be frequently useful in geometrical as well as in physical inquiries. If we investigate the equations (12), and have regard in so doing to the present variability of $x_{,}, y_{,}, z_{,}$, we shall find (23)

$$\begin{aligned} \delta x_{1} - \delta x_{,} &= \delta\xi_{1} - y_{,}\delta R + z_{,}\delta Q, \\ \delta y_{1} - \delta y_{,} &= \delta\eta_{1} - z_{,}\delta P + x_{,}\delta R, \\ \delta z_{1} - \delta z_{,} &= \delta\zeta_{1} - x_{,}\delta Q + y_{,}\delta P. \end{aligned}$$

But if we add equations (9) together after multiplying them respectively by $L_{\prime}$, $M_{\prime}$, and $N_{\prime}$, and reduce by means of (20), we shall find

$$L_{\prime}\delta x_{1} + M_{\prime}\delta y_{1} + N_{\prime}\delta z_{1} = L\delta x + L'\delta x' + L''\delta x'';$$

which the equations of the two surfaces enable us to write in this form,

$$L_{\prime}(\delta x_{1} - \delta x_{\prime}) + M_{\prime}(\delta y_{1} - \delta y_{\prime}) + N_{\prime}(\delta z_{1} - \delta z_{\prime}) = 0\,.$$

This last expression becomes, with the aid of the three equations given above,

$$\left.\begin{array}{l} L_{\prime}(\delta\xi_{1} - y_{\prime}\delta R + z_{\prime}\delta Q) \\ M_{\prime}(\delta\eta_{1} - z_{\prime}\delta P + x_{\prime}\delta R) \\ N_{\prime}(\delta\zeta_{1} - x_{\prime}\delta Q + y_{\prime}\delta P) \end{array}\right\} = 0,$$

which is the same with the result before obtained, and is in fact expressing algebraically that the velocity of the point of contact is *nought* in the direction of the normal. Care must be taken to distinguish between δx_{1}, δy_{1}, δz_{1}, and $\delta x_{\prime}$, $\delta y_{\prime}$, $\delta z_{\prime}$. They both denote the variations of the point of contact (P) estimated in the direction of the body-axes; but the former denote its variation along the surface of the moving body, the latter its variation along the surface of support. The former are of the kind called incomplete variations, the body axes being supposed to remain fixed *during* any one of these variations, and to vary instantaneously in passing to the next. The latter are the total variations of the actual body coordinates of the point of contact (P). These two kinds of variations never coincide in value except in the case of rolling motion unaccompanied by sliding.

If the body is supported upon two, three, four or five given surfaces, there will be as many equations of condition similar to equation (22) as there are surfaces of support: if the body is required to be in contact with six given surfaces, its station and aspect become determinate and motion is no longer possi-

ble; the formulas I have given will however be very useful in investigating the position of the body. If there be absolutely no friction, in that case the above equations of condition are the only ones which exist along with the general dynamical equation. But if there be proposed any hypothesis of friction or analogous restraint, the following considerations will assist us in determining the relations between the momentary changes in translation and rotation.

Let us, for the sake of greater generality, suppose that the two bodies M and M', which are in contact with each other, are both of them in motion. There will be now at least *six* different velocities at the point of contact, liable without attention to be confounded with each other:—I. The absolute velocity in space of the physical point of contact p of the body M. II. The absolute velocity in space of the physical point of contact p' belonging to the body M'. III. The absolute velocity of the geometrical point of contact P. IV. The velocity with which the point P changes its place on the surface of M. V. The velocity with which the same point P changes its place on the surface of the body M'. VI. The velocity of rasure.—The same distinctions are to be observed with respect to the directions which belong to these velocities. The effects of friction at the point of contact will depend entirely upon the velocity and direction of rasure, which are the same with which the physical points p and p' recede from each other in the instant after they meet at the geometrical point P. If one of the bodies as M' be fixed, then this velocity and direction will be the same with the absolute velocity and direction of the physical point p, and the velocities of rasure in the direction of the body coordinates will therefore be denoted generally by

$$\begin{aligned} &d\xi_1 - y_, dR + z_, dQ, \\ &d\eta_1 - z_, dP + x_, dR, \\ &d\zeta_1 - x_, dQ + y_, dP, \end{aligned}$$

or, when necessary, by the values (23), which we have shewn to be equivalent to the above expressions. We may suppose

the friction to be a function of these velocities or of the pressure or of both conjointly. The effect of this would be to add to the other accelerating forces three new ones applied to the point $(x_, y_, z_,)$ of the form of

$$\phi\frac{dx_1 - dx_,}{dv}, \qquad \phi\frac{dy_1 - dy_,}{dv}, \qquad \phi\frac{dz_1 - dz_,}{dv},$$

where ϕ is any given function of the pressure and velocity of rasure dv, dv itself being equal to

$$\sqrt{[(dx_1 - dx_,)^2 + (dy_1 - dy_,)^2 + (dz_1 - dz_,)^2]}\,.$$

The pressure is then to be eliminated from the equations of motion; after which there will remain a number of equations sufficient, in conjunction with the equations of the surfaces, to determine the position of the body in terms of the time.

If the friction, be the cause of it what it may, be exactly sufficient to prevent all sliding, while it offers no impediment to the body's revolution round the normal at the point of contact, the motions will be of a nature much more resembling actual oscillations and rotations on supporting surfaces, than in the hypothesis of surfaces absolutely smooth, particularly when the tangent plane at P remains throughout the motion nearly horizontal. The effects of this kind of motion, of which the pendulum with a cylindrical axis is the simplest possible species, have not, that I know of, been examined by any author, when the triple rotation of pitching, rocking and whirling are all considered at once. Nevertheless, the problem of the small oscillations of the kind above described upon a plane or spherical surface is susceptible of complete integration and solution in the case both of free sliding and perfect rolling, whatever be the figure and constitution of the oscillating body, and whatever be the velocity round one of the axes, provided that it be compatible with small rotations round the other two. I have given in the New York Mathematical Diary for July 1827 formulas which are applicable to the case of all bodies,

of any shape and density whatever with a spherical areola of contact, whirling and oscillating with a perfect rolling motion on an horizontal plane. The method I now offer is intended to comprise every form of this areola, having regard at the same time to the nature of the surface of support.

When the friction prevents all sliding, the elements of the curves described on the two surfaces are equal, and moreover coincide at every instant of the arbitrary variations, so that we have necessarily (24)

$$\begin{aligned} \delta x &= a\,\delta x_{,} + b\,\delta y_{,} + c\,\delta z_{,}, \\ \delta x' &= a'\delta x_{,} + b'\delta y_{,} + c'\delta z_{,}, \\ \delta x'' &= a''\delta x_{,} + b''\delta y_{,} + c''\delta z_{,}. \end{aligned}$$

These values reduce equations (21) to

$$\begin{aligned} 0 &= \delta\xi + x_{,}\delta a + y_{,}\delta b + z_{,}\delta c\,, \\ 0 &= \delta\xi' + x_{,}\delta a' + y_{,}\delta b' + z_{,}\delta c'\,, \\ 0 &= \delta\xi'' + x_{,}\delta a'' + y_{,}\delta b'' + z_{,}\delta c''; \end{aligned}$$

or, substituting for the variations of the cosines their values as given by equations (6),

$$\begin{aligned} 0 &= \delta\xi + (c\,y_{,} - b\,z_{,})\delta P + (a\,z_{,} - c\,x_{,})\delta Q + (b\,x_{,} - a\,y_{,})\delta R, \\ 0 &= \delta\xi' + (c'y_{,} - b'z_{,})\delta P + (a'z_{,} - c'x_{,})\delta Q + (b'x_{,} - a'y_{,})\delta R, \\ 0 &= \delta\xi'' + (c''y_{,} - b''z_{,})\delta P + (a''z_{,} - c''x_{,})\delta Q + (b''x_{,} - a''y_{,})\delta R; \end{aligned}$$

expressions which, by means of equations (10) and the reductions arising from the relations (4), may be presented in this form (25)

$$\begin{aligned} 0 &= \delta\xi_{1} + z_{,}\delta Q - y_{,}\delta R, \\ 0 &= \delta\eta_{1} + x_{,}\delta R - z_{,}\delta P, \\ 0 &= \delta\zeta_{1} + y_{,}\delta P - x_{,}\delta Q. \end{aligned}$$

These are the relations which the condition of the peculiar

motion now considered introduces among the variations of the six elements of position*.

The same results may be obtained immediately from the equation $\delta v = 0$ (which is the fundamental equation of this kind of motion) taken in connection with the value of δv given above. It is evident that we have also in this case

$$\delta x_1 = \delta x_{,}, \qquad \delta y_1 = \delta y_{,}, \qquad \delta z_1 = \delta z_{,}.$$

If there be a second surface of support upon which also the body is to roll without sliding, we shall have three other equations exactly similar to the above. If we denote by $\delta\alpha_1$, $\delta\beta_1$, $\delta\gamma_1$ the variations in the direction of the body-axes of a fixed point in the body whose body-coordinates referred to $O_{,}$ are α, β, γ, we shall obtain from equations (12)

$$\begin{aligned}
\delta\alpha_1 &= \delta\xi_1 + \gamma\delta Q - \beta\delta R,\\
\delta\beta_1 &= \delta\eta_1 + \alpha\delta Q - \gamma\delta P,\\
\delta\gamma_1 &= \delta\zeta_1 + \beta\delta Q - \alpha\delta Q.
\end{aligned}$$

From which if we subtract (25) we have

$$\begin{aligned}
\delta\alpha_1 &= (\gamma - z_{,})\delta Q - (\beta - y_{,})\delta R,\\
\delta\beta_1 &= (\alpha - x_{,})\delta R - (\gamma - z_{,})\delta P,\\
\delta\gamma_1 &= (\beta - y_{,})\delta P - (\alpha - x_{,})\delta Q.
\end{aligned}$$

For the points of the body which are momentarily at rest, both sides of these equations become equal to nought, and we obtain

$$\frac{\alpha - x_{,}}{dP} = \frac{\beta - y_{,}}{dQ} = \frac{\gamma - z_{,}}{dR},$$

the equation of a straight line passing through the point of contact and parallel to the axis of instantaneous rotation.

* Since this communication was handed to the Librarian to be read before the Society, Mr E. Nulty has shown me the above three formulas, derived (in the solution of a problem that had recently occupied his attention) from the ingenious consideration that in perfect rolling the motion of the physical point of contact in the direction of the body-axes is equal and opposite to the motion of the point in which these axes are supposed to have their origin.

When the surfaces are considered as perfectly smooth, we have seen that there are as many equations of condition as there are surfaces of support to be taken in conjunction with the general dynamical equation. Multiplying each of these equations by an indeterminate coefficient and equating to nought the sums of the coefficients of the variations, there results (26)

$$d^2\xi_2 + SX_{\prime}Dm + \Sigma\theta L_{\prime} = 0,$$
$$d^2\eta_2 + SY_{\prime}Dm + \Sigma\theta M_{\prime} = 0,$$
$$d^2\zeta_2 + SZ_{\prime}Dm + \Sigma\theta N_{\prime} = 0;$$

$$U + S(Z_{\prime}y_{\prime} - Y_{\prime}z_{\prime})Dm + \Sigma\theta(N_{\prime}y_{\prime} - M_{\prime}z_{\prime}) = 0,$$
$$V + S(X_{\prime}z_{\prime} - Z_{\prime}x_{\prime})Dm + \Sigma\theta(L_{\prime}z_{\prime} - N_{\prime}x_{\prime}) = 0,$$
$$W + S(Y_{\prime}x_{\prime} - X_{\prime}y_{\prime})Dm + \Sigma\theta(M_{\prime}x_{\prime} - L_{\prime}y_{\prime}) = 0:$$

where Σ denotes the sum of similar quantities, θ one of the indeterminate coefficients, the mass at the same time being put equal to unity.

These equations are evidently the same as those which would have been obtained immediately by substituting in place of the surfaces unknown forces acting constantly in the direction of the normals at the variable points of contact, and then considering the system as free. The equations of condition however would still have been indispensable, in order to supply the number of equations lost in the elimination of the unknown forces of reaction. I should also on other accounts have preferred investigating these equations by the preceding method; because it furnishes a variety of formulas useful in the analytical geometry of touching surfaces, and extremely convenient in the determination of the motions of bodies subject to a friction producing some assignable relation between their sliding and their rolling motions.

If we restrict ourselves to the examination of the motion on a single surface, the body being acted on by common gravity g, the preceding formulas become, (reckoning the positive

coordinates x'' downward from the horizontal plane of x and x' and observing that we have

$$\begin{array}{lll} X_{,} = -ga'', & Y_{,} = -gb'', & Z_{,} = -gc'', \\ Sx_{,}Dm = 0, & Sy_{,}Dm = 0, & Sz_{,}Dm = 0, \end{array}$$

the other quantities remaining as before,)

$$\begin{aligned} d^2\xi_2 + \theta L_{,} &= ga'', \\ d^2\eta_2 + \theta M_{,} &= gb'', \\ d^2\zeta_2 + \theta N_{,} &= gc''; \end{aligned}$$

$$\begin{aligned} U + \theta(N_{,}y_{,} - M_{,}z_{,}) &= 0, \\ V + \theta(L_{,}z_{,} - N_{,}x_{,}) &= 0, \\ W + \theta(M_{,}x_{,} - L_{,}y_{,}) &= 0: \end{aligned}$$

from which θ being eliminated, there will remain five equations, which along with the equation of condition comprehend and determine all the phenomena of the motion. The first three of the above equations may by means of formulas (8) and (11) be presented in this form (28)

$$\begin{aligned} d^2\xi + \theta L &= 0, \\ d^2\xi' + \theta L' &= 0, \\ d^2\xi'' + \theta L'' &= g; \end{aligned}$$

which are in appearance simpler than the others, more especially as the accelerations are now complete. It will however be found necessary to have recourse to the former, except when the supporting surface is a plane, or the supported body is a homogeneous sphere.

Let us now suppose that the surface of the moving body and the surface of support are both of the second degree. For the sake of greater simplicity, let us suppose also that the rectangular diameters of the surface of support coincide with the axes in space, and that the centre of the moving body when

it has a centre, or the summit or a point in the axis when it is without a centre, is at the same time its centre of gravity. Let the equations of the two surfaces be respectively

$$\sqrt{\left(\frac{x^2}{\alpha^2}+\frac{x'^2}{\alpha'^2}+\frac{x''^2}{\alpha''^2}\right)}-1=0,$$

$$\sqrt{\left(\frac{x_{\prime}^2}{\alpha_{\prime}^2}+\frac{y_{\prime}^2}{\beta_{\prime}^2}+\frac{z_{\prime}^2}{\gamma_{\prime}^2}\right)}-1=0,$$

where the constants are the semi-axes of the figure. Or, what will be more commodious in the present instance, let these equations be presented in the forms

$$\sqrt{(A x^2+A' x'^2+A'' x''^2)}-1=0,$$
$$\sqrt{(A_{\prime} x_{\prime}^2+B_{\prime} y_{\prime}^2+C_{\prime} z_{\prime}^2)}-1=0,$$

where the constants are the reciprocals of the squares of the semi-axes, and A of course not to be confounded with the A used before. These equations, although apparently only intended for ellipsoids, spheroids of revolution, and spheres, will answer for all surfaces of the second degree whatever, provided the following changes be made in the results to which the above would lead.

1. For a single-napped hyperboloid, change the sign of the square of the semi-axis of the ellipsoid corresponding to the imaginary axis.

2. For a double-napped hyperboloid, change the signs of the squares of the two semi-axes of the ellipsoid which correspond to the two imaginary axes.

3. For an elliptical paraboloid, diminish, in the results, the coordinates parallel to the figure's axis by the corresponding semi-axis of the ellipsoid; then make all the semi-axes infinite, but so that the two third proportionals to the first mentioned semi-axis and each of the other two, shall remain finite and

be equal to the semi-parameters of the principal parabolic sections.

4. For a hyperbolic paraboloid, the same transformation, changing the sign of the parameter of the principal negative parabola. The origin of the coordinates of the paraboloids will then be at the summit of the axis.

5. For an elliptical or circular cone, change the sign of the square of the semi-axis corresponding to the axis of the cone; then make all the semi-axes infinite, but so that that the ratios of the semi-axis first mentioned to the other two may be equal to the ratios of any altitude of the cone to the semi-axes of the corresponding base.

6. For an elliptical or circular cylinder, make infinite the semi-axis of the ellipsoid corresponding to the infinite axis of the cylinder.

7. For a hyperbolic cylinder, make a similar alteration, and change the sign of the square of the semi-axis which corresponds to the imaginary axis of the principal hyperbolic section.

8. For a parabolic cylinder, the same alterations as for either of the paraboloids, making infinite at the same time the third proportional to the two semi-axes corresponding to the normal and the infinite axes of the cylinder.

The values of the cosines of the normal's axe-angles obtained by means of the differential formulas (19) lead to the following equations: (29)

$$\begin{aligned} L &= kA\,x\,, & L_{,} &= k_{,}A_{,}x\,, \\ L' &= kA'x'\,, & M_{,} &= k_{,}B_{,}y\,, \\ L'' &= kA''x''\,, & N_{,} &= k_{,}C_{,}z\,; \end{aligned}$$

where k and $k_{,}$ are respectively equal to

$$\frac{1}{\sqrt{(A^2x^2+A'^2x'^2+A''^2x''^2)}}\,, \qquad \frac{1}{\sqrt{(A_{,}^2x_{,}^2+B_{,}^2y_{,}^2+C_{,}^2z_{,}^2)}}\,.$$

The above equations may be so combined with the equations of the surface, as to furnish other forms for k and $k_{,}$ namely

$$\sqrt{(\alpha^2 L^2 + \alpha'^2 L'^2 + \alpha''^2 L''^2)}, \qquad \sqrt{(\alpha_,^2 L_,^2 + \beta_,^2 M_,^2 + \gamma_,^2 N_,^2)}.$$

Finally, it is easy to verify the following values of these same quantities:

$$k = Lx + L'x' + L''x'', \qquad k_, = L_,x_, + M_,y_, + N_,z_,.$$

These last expressions are susceptible of an obvious geometrical interpretation, and show that k and $k_,$ are the projections of the two radius vectors of the point of contact upon the common normal at that point.

The conditions (20) of a common normal moreover give (30)

$$\begin{aligned}
k\mathcal{A}\,x &= k_,(\mathcal{A}_,a\,x_, + B_,b\,y_, + C_,c\,z_,),\\
k\mathcal{A}'x' &= k_,(\mathcal{A}_,a'x_, + B_,b'y_, + C_,c'z_,),\\
k\mathcal{A}''x'' &= k_,(\mathcal{A}_,a''x_, + B_,b''y_, + C_,c''z_,);
\end{aligned}$$

$$\begin{aligned}
k_,\mathcal{A}_,x_, &= k(\mathcal{A}\,a\,x + \mathcal{A}'a'x' + \mathcal{A}''a''x''),\\
k_,B_,y_, &= k(\mathcal{A}\,b\,x + \mathcal{A}'b'x' + \mathcal{A}''b''x''),\\
k_,C_,z_, &= k(\mathcal{A}\,c\,x + \mathcal{A}'c'x' + \mathcal{A}''c''x'').
\end{aligned}$$

From each of these triplets may be obtained expressions for the ratio of the two projections which may occasionally be useful. If we add the first three equations together after multiplying the first equation by x, the second by x', and the third by x'', reducing by means of equations (4) and the equation of the surface of support, and proceed by an analogous method with the second triplet, employing in the reduction the surface of the moving body, we shall obtain the two equations

$$\frac{k}{k_,} = (x_, + \xi_,)\frac{x_,}{\alpha_,^2} + (y_, + \eta_,)\frac{y_,}{\beta_,^2} + (z_, + \zeta_,)\frac{z_,}{\gamma_,^2},$$

$$\frac{k_,}{k} = (x - \xi)\frac{x}{\alpha^2} + (x' - \xi')\frac{x'}{\alpha'^2} + (x'' - \xi'')\frac{x''}{\alpha''^2},$$

which, by means of the equations of the surfaces, will become

$$\frac{k}{k_{,}} = 1 + \left(\frac{x_{,}\xi_{,}}{\alpha_{,}^{2}} + \frac{y_{,}\eta_{,}}{\beta_{,}^{2}} + \frac{z_{,}\zeta_{,}}{\gamma_{,}^{2}}\right),$$

$$\frac{k_{,}}{k} = 1 - \left(\frac{x\xi}{\alpha^{2}} + \frac{x'\xi'}{\alpha'^{2}} + \frac{x''\xi''}{\alpha''^{2}}\right).$$

If now we substitute the first of these two values in the first triplet of equations (30) and then substitute the values of x, x', x'' thus transformed in the first triplet of equations (2), $x_{,}$, $y_{,}$, $z_{,}$ may be determined by quadratics in terms of $\xi_{,}$, $\eta_{,}$, $\zeta_{,}$ and the aspect of the body. By a process altogether similar, x, x', x'' may be obtained in terms of ξ, ξ', ξ''. At the same time it ought to be observed, that whatever be the nature of the surfaces, if from the seven equations $K = 0$, $K_{,} = 0$, either of the triplets (2), and any two of the three equations of contact (20) (the three being in fact equivalent to two in consequence of the condition $L^{2} + L'^{2} + L''^{2} = L_{,}^{2} + M_{,}^{2} + N_{,}^{2}$) we eliminate the space and body coordinates of the point of contact, there will remain an equation of condition between the elements of the station and the aspect of the body, of which equation (22) is in all cases the differential.

As the angular velocities p, q, r are functions of the nine cosines and their differentials, and as these are connected by six equations of condition and variously expressible in functions of the three elements of the aspect of the body, it follows that the six equations of motion will by the above mentioned substitutions involve, beside the time, the six elements of the position of the body.

If we substitute in place of $L_{,}$, $M_{,}$, $N_{,}$ in the equations of motion their values as given by equations (29) and employ at the same time the abridgments,

$$C_{,} - B_{,} = A_{2}, \qquad A_{,} - C_{,} = B_{2}, \qquad B_{,} - A_{,} = C_{2},$$

we shall find (31)

$$\begin{aligned} d^2\xi_2 + A_, \theta k_, x_, &= ga'', \\ d^2\eta_2 + B_, \theta k_, y_, &= gb'', \\ d^2\zeta_2 + C_, \theta k_, z_, &= gc''; \end{aligned}$$

$$\begin{aligned} U + A_2 \theta k_, y_, z_, &= 0, \\ V + B_2 \theta k_, z_, x_, &= 0, \\ W + C_2 \theta k_, x_, y_, &= 0: \end{aligned}$$

equations to which we shall have occasion to refer hereafter.

To return now for a moment to the general problem. If we add the second triplet of equations (27) together, after multiplying the first equation by $L_,$, the second by $M_,$ and the third by $N_,$, there results

$$L_, U + M_, V + N_, W = 0.$$

Substituting in this equation, in place of the six quantities which it involves, their values (15) and (20), and reducing by means of the relations (4), we shall find

$$\left.\begin{aligned} L\,.\,d(a\,P' + b\,Q' + c\,R') \\ L'.\,d(a'P' + b'Q' + c'R') \\ L''.\,d(a''P' + b''Q' + c''R') \end{aligned}\right\} = 0,$$

where P', Q', R' are the partial differentials, with respect to p, q, r, of the function

$$T = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2) - (Fqr + Grp + Hpq),$$

which is one half of the living forces of the body arising from its motion of rotation.

By means of equations (20), the relations (4) and the substitutions (6), it will be found that the foregoing equation is susceptible of being presented in the following form:

$$d(L_{,}P' + M_{,}Q' + N_{,}R') = \begin{cases} P'(dL_{,} + M_{,}dR - N_{,}dQ) \\ Q'(dM_{,} + N_{,}dP - L_{,}dR) \\ R'(dN_{,} + L_{,}dQ - M_{,}dP). \end{cases}$$

These equations are true of all supporting and supported surfaces whatever. It might easily be shown that this last equation is capable of being derived from the principle that the rate of increase of the sum of all the areas projected on the plane tangent to the point of variable contact is momentarily constant, the tangent plane being supposed to remain for a moment fixed while the body passes on to its consecutive position on the surface of support.

When the sustaining surface is an inclined plane, L, L', L'' become constant, and the right member of the last equation will vanish on the substitution of the values which $L_{,}$, $M_{,}$, N acquire in such a case, so that the equation becomes integrable with respect to time, and we obtain

$$L_{,}P' + M_{,}Q' + N_{,}R' = I,$$

I being an arbitrary constant.

Again, if we add together the second triplet (27), after multiplying the three equations respectively by dP, dQ, dR, and reduce by means of the equation of condition (22), we obtain

$$UdP + VdQ + WdR - \theta(Ld\xi + L'd\xi' + L''d\xi'') = 0.$$

Substituting for L, L', L'' their values (20), and performing the operation indicated in the first three terms of this equation, there will result

$$dT + d\xi d^2\xi + d\xi' d^2\xi' + d\xi'' d^2\xi'' - gd\xi'' = 0,$$

an equation whose integral gives us the principle of living for-

ces applied to the problem of any solid body rolling on any given surface,

$$T + \tfrac{1}{2}(d\xi^2 + d\xi'^2 + d\xi''^2) - g\xi'' = J,$$

J being another arbitrary constant, and $M = 1$.

It is evident, moreover, that the same triplet furnishes the relation

$$Ux_{,} + Vy_{,} + Wz_{,} = 0.$$

When the body is in a state of permanent equilibrium upon the surface of support, the velocities and accelerations of the six elements of its position are nought, and the six equations of motion give us

$$\begin{aligned} L_{,} &= a'', \qquad & N_{,}y_{,} - M_{,}z_{,} &= 0, \\ M_{,} &= b'', \qquad & L_{,}z_{,} - N_{,}x_{,} &= 0, \\ N_{,} &= c''; \qquad & M_{,}x_{,} - L_{,}y_{,} &= 0. \end{aligned}$$

The first equations express that the direction of the normal is vertical, the others that it passes through the centre of gravity. In general we may observe, that the equations of the motion of rotation are in fact the equations of the normal at the point of contact, and that the distance of the normal from the centre of gravity is at all times equal to

$$\frac{\sqrt{(U^2+V^2+W^2)}}{\theta};$$

so that $\sqrt{(U^2 + V^2 + W^2)}$ represents the effect which the plane's reaction on the body has in producing the motion of rotation. The line which passes through the centre of gravity and any one of the points $B_{,}$ of the surface on which it may be balanced is not in general a principal axis; but as the preceding formulas are independent of the position of these axes, we are

permitted to take any of the lines $O_{,}B_{,}$ for the axis of $z_{,}$. For the sake of greater brevity we may call the points $B_{,}$ the *ballancing points* and the lines $O_{,}B_{,}$ the *natural verticals* of the body.

The phenomena of the motions of the body immediately about its state of equilibrium will manifestly depend upon the configuration of the surfaces or *areolas* as we may term them in the immediate vicinities of the two points B and $B_{,}$, the former denoting any of the points of the sustaining surface with which $B_{,}$ may be in contact when the body is at rest. From the established theory of contacts, it follows that every point, not singular, of any surface whatever may be brought into a contact of the second order with some curve surface of the second degree. Dupin, in particular, has shewn, in his excellent supplement to the *Analytical Geometry* of Monge, that every plane section of any curve surface parallel to a tangent plane and infinitely near to it is a conic section, indicating all the characters of the curvature around the point touched by the tangent plane. It is easy to infer from this, that for all phenomena depending upon the curvature of the areolas at B and $B_{,}$ these points may in all cases be regarded as the *summits of paraboloids*, elliptical, hyperbolical or intermediate. This proposition, which is fundamental, might be also proved thus. Let x, y, z denote the coordinates of either areola reckoning from B or $B_{,}$ along the tangent plane and normal. The most general equation of the areola will then be

$$z = \mathfrak{A}x^2 + Bxy + Cy^2,$$

the condition of a tangent plane requiring that z should be of two dimensions in x and y, and the condition that the point is not a singular one excluding fractional and negative exponents. As the direction of the axes x and y in the tangent plane is arbitrary, the term Bxy may be made to disappear, and the equation becomes simply

$$z = \mathfrak{A}'x^2 + C'y^2,$$

a paraboloid, elliptical, hyperbolical or intermediate, according as C' is positive, negative or nought, the constants A' and C' representing the reciprocals of the greatest and least diameters of curvature. In a similar manner it might be shown that every areola whatever may be represented by the areola around the summit of some assignable hyperboloid with an arbitrary vertical axis,—elliptical when the areola is *concurvate*, that is with the curvatures of all its normal sections directed the same way,—hyperbolical when *discurvate*, or with the curvatures of its normal sections directed some one way and some the opposite,—cylindrical when the curvature of the areola is intermediate as in the case of developable surfaces.

It follows therefore, from what precedes, that in the problem of the small oscillations of supported bodies, the equations (31) obtained above for surfaces of the second degree, with the positions there proposed, will answer for all possible areolas of contact, the arbitrary values of the axes α'' and $\gamma_{\prime}$ enabling us to avail ourselves completely of this simplification by placing the centre of the osculating figure in the centre of gravity of the body, at the same time that we may take any point at pleasure in the vertical through B for the origin of the invariable axes.

The hypothesis that, during the motion of the body, its natural vertical declines but very little from the position which it would occupy if at rest, is equivalent to supposing that c and c' are at all times very small, and we shall regard them therefore in the following calculations as infinitesimals of the first order. The hypothesis that the two areolas of contact are indefinitely small is analytically expressed by considering x and x', $x_{\prime}$ and $y_{\prime}$ as quantities infinitely small. The preceding formulas will now enable us to ascertain what values the rest of the denoted quantities acquire in consequence of these two hypotheses, and the conditions of their legitimacy will appear in the equations of condition which arise in the course of the solution of the problem. The fundamental relations (4) give us in the first place, neglecting all infinitesimals of higher orders than the first, $c'' = 1$, $b = -a'$, $b' = a$. The values of

p, q and r are best obtained by means of formulas (6). They furnish immediately (32)

$$p = ra'' + db'', \qquad\qquad q = rb'' - da''.$$

The same equations give $da = bdR$, $db = -adR$, which, integrated in conjunction with $a^2 + b^2 = 1$, give us $a = \cos R$, $b = -\sin R$, the angle R being counted from the axis of x. The nine cosines then become

$$\begin{array}{lll} a = \cos R, & b = -\sin R, & c = b''\sin R - a''\cos R, \\ a' = \sin R, & b' = \cos R, & c' = -b''\cos R - a''\sin R, \\ a'' = a''; & b'' = b''; & c'' = 1. \end{array}$$

From equations (8) and the equations of the surfaces we obtain

$$\begin{array}{ll} x'' = \alpha'', & \xi'' = \alpha'' - \gamma_{,}, \\ z_{,} = \gamma; & \zeta' = \alpha'' - \gamma_{,}. \end{array}$$

The analysis gives these constants the double sign, which I omit, as in case of application it will always be immediately obvious which will be affected with $+$ and which with $-$. Thus if both areolas are concave upward, and the centre of gravity of the oscillating body is above the point of contact and below the centre of the figure which osculates with the areola of support, then the signs remain as above, the ellipsoid or elliptical paraboloid being in such a case the proper osculating figure. If, as in the common pendulum, the point $O_{,}$ is below $B_{,}$, and the two areolas are still concave upward, the osculatrix of the areola at $B_{,}$ must be an hyperboloid or elliptical paraboloid with the point $O_{,}$ taken in the prolongation of the axis, and the constant α'' would change its sign. If the pendulum were hung upon a fixed *annulus* interlinking with another annulus at the upper extremity of the pendulum, both areolas would then become *discurvate* and the osculating figures would be either single-napped hyperboloids or hyper-

bolic paraboloids. In cases of this kind, it may be well to observe at once, the analysis does not necessarily regard the motion round the normal as arrested by the impenetrability of the rings, but implies in general a mutual penetrability so as to admit but a single point of contact.

The law of continuity, a law to which analysis, in all its processes, adheres with singular and sometimes indeed with inconvenient faithfulness, requires us to attribute to *both sides* of the supporting surface the power of feeling and sustaining in *both directions*, the presence of the moving body. Thus, if we suppose a sphere in motion on the *outside* of another sphere, it would evidently come, at some determinate epoch, into a position where its pressure on the supporting surface would be nought. It would there leave the surface, and its motion afterwards would be a separate problem. An analytical solution of the question however would regard the moving body as still connected with the surface of support, and exerting on it a pressure tending to draw it outward from its centre. This pressure would be such as would arise from a momentary but continually renewed connecting thread infinitely short passing from sphere to sphere at the point of variable contact, or such as would take place if we supposed the surfaces of one of the spheres to consist of two concentric spherical surfaces infinitely near each other, and the momentary point of contact of the other sphere to be always engaged and confined between them. Again, let us suppose that a circle rolls and slides inside down an ellipsis whose maximum curvature is greater and whose minimum is less than the curvature of the circle. If we suppose moreover the long axis vertical and the short axis longer than the diameter of the circle, the circle in descending will come first to a place where it will touch the ellipsis in two points and there physically it would stop, but the analysis (on the hypothesis of one original point of contact) will consider the circle as geometrical except at this point of contact, and of course will represent the circle as passing onward unimpeded by this second contact. It will then reach a point in the ellipse where the

curvatures of the two curves are equal, and where on one side of the point of osculation the circle passes inside, and on the other outside of the ellipsis. Before the circle comes into this position the arc of contact is entirely within, after it leaves it entirely without, the ellipsis, and the connection must be maintained as in the preceding example. The same remarks will apply to the motion of an ellipsoid placed within a sphere of a curvature intermediate between the greatest and least curvature of the ellipsoid, to all contacts between discurvate surfaces, and in general to all cases in which the *maximum* curvature of one of the surfaces is not less than the *minimum* curvature of the other.

In order to determine the actual oscillatory motions of such bodies, we must institute as many equations of condition similar to (22) as the moving body can have points of contact with the supporting surface. We must then determine when the pressure at any one of these points becomes equal to nought, after which the problem is to be considered as a new one, and the subsequent motion of the body must be traced by applying to it the equations resulting from one contact less than before, until the body either again comes into a fresh point of contact, or loses another of the contacts which it was supposed to have at first. In the course of the various positions into which the moving body would come, it would frequently happen that two of the points would unite into one by an inosculation of the curves of contact, or one would become two, as when a sphere moves upon an oval annulus of smaller dimensions than the sphere from the concurvate to the discurvate portion of it. An inquiry into motions of this kind is however foreign to the purpose of this paper, and I return to the consideration of the problem when restricted to a single point of contact.

The selection of a paraboloid, in its three varieties of elliptical, hyperbolical and intermediate, to serve as the osculating figure of the areola at the balancing point of the body, is attended with the advantage that, beside suiting all possible cases of curvature, it is always applicable, whether the centre of

gravity be *at* the balancing point, *above* it, or *below* it. This is evident from the equation of the curve,

$$\frac{x_,^2}{\alpha_,} + \frac{y_,^2}{\beta_,} + 2(z_, - \gamma_,) = 0,$$

where it is manifest that $\gamma_,$ may be taken arbitrarily, positive, negative or nought, without producing any other change than an elevation or depression of the origin, while the different values and signs which we may ascribe to $\alpha_,$ and $\beta_,$ will furnish us with areolas of every variety of curvature. This advantage is however unimportant in the present inquiry, which is rather to ascertain the results of the general problem than to enter into a detailed examination of each particular case. Resuming therefore the expressions (30) before obtained for ellipsoids on ellipsoidal surfaces, and observing that the quantities k and $k_,$ in the case of small oscillations become constant and equal to the fixed and moveable vertical semi-axes, retaining at the same time the symbols c'', x'', $z_,$, ξ'', $\zeta_,$, in order to permit without further substitutions the application of the usual formulas, the second triplet of equations (30) furnish, when the areola of support is spherical, whatever be the form of the areola around the balancing point of the oscillating body,

$$\begin{aligned} k_, A_, x_, &= kA(ax + a'x' + a''x''), \\ k_, B_, y_, &= kA(bx + b'x' + b''x''), \\ k_, C_, z_, &= kA(cx + c'x' + c''x''). \end{aligned}$$

By means of equations (3) these become

$$\begin{aligned} \alpha\gamma_, A_, x_, &= x_, + \xi_,, \\ \alpha\gamma_, B_, y_, &= y_, + \eta_,, \\ \alpha\gamma_, C_, z_, &= z_, + \zeta_,; \end{aligned} \qquad \text{or} \qquad \begin{aligned} x_, &= l\xi_,, \\ y_, &= m\eta_,, \\ z_, &= n\zeta_,. \end{aligned}$$

Substituting the values of $x_,$, $y_,$, $z_,$ in equations (31) and employing the following abridgments,

$$\frac{1}{A_{,}lk_{,}} = \alpha - \frac{\alpha_{,}^{2}}{\gamma_{,}} = \lambda, \qquad \frac{1}{B_{,}mk_{,}} = \alpha - \frac{\beta_{,}^{2}}{\gamma_{,}} = \mu,$$

$$B_{2}lk_{,}\eta\zeta_{,} = \frac{\gamma_{,}^{2} - \alpha_{,}^{2}}{\alpha\gamma_{,} - \alpha_{,}^{2}} = A_{,,}; \qquad A_{2}mk_{,}\eta\zeta_{,} = \frac{\gamma_{,}^{2} - \beta_{,}^{2}}{\alpha\gamma_{,} - \beta_{,}^{2}} = B_{,,},$$

omitting infinitesimals of the second order, and restoring dt and M, we obtain (34)

$$\frac{d^{2}\xi_{2}}{dt^{2}} + \frac{\theta}{\lambda}\xi_{,} = ga'',$$

$$\frac{d^{2}\eta_{2}}{dt^{2}} + \frac{\theta}{\mu}\eta_{,} = gb'',$$

$$\theta = g;$$

$$\begin{aligned} U - M\theta B_{,,}\eta_{,} &= 0, \\ V + M\theta A_{,,}\xi_{,} &= 0, \\ W &= 0. \end{aligned}$$

By an examination of the values of the first and second differentials of the indefinite integrals ξ_{1}, η_{1}, ζ_{1}, ξ_{2}, η_{2}, ζ_{2} given by equations (10), it will readily be seen that, with the assistance of the relations (4), (6), (8), the following expressions will be verified (35)

$$\begin{aligned} d\xi_{1} &= d\xi_{,} - \eta_{,}dR + \zeta_{,}dQ, \\ d\eta_{1} &= d\eta_{,} - \zeta_{,}dP + \xi_{,}dR, \\ d\zeta_{1} &= d\zeta_{,} - \xi_{,}dQ + \eta_{,}dP; \end{aligned}$$

$$\begin{aligned} d^{2}\xi_{2} &= d^{2}\xi_{1} - d\eta_{1}dR + d\zeta_{1}dQ, \\ d^{2}\eta_{2} &= d^{2}\eta_{1} - d\zeta_{1}dP + d\xi_{1}dR, \\ d^{2}\zeta_{2} &= d^{2}\zeta_{1} - d\xi_{1}dQ + d\eta_{1}dP: \end{aligned}$$

equations analogous to those first obtained by Lagrange to denote the motions of rotation of a system of particles which have at the same time individual motions of their own. In the case of small oscillations the third and sixth of these equa-

tions vanish altogether, as all the terms are infinitesimals of the second order, and the other four become (36)

$$d\xi_1 = d\xi_{,} - \eta_{,} dR + \zeta_{,} dQ,$$
$$d\eta_1 = d\eta_{,} - \zeta_{,} dP + \xi_{,} dR;$$

$$d^2\xi_2 = d^2\xi_1 - d\eta_1 dR,$$
$$d^2\eta_2 = d^2\eta_1 + d\xi_1 dR:$$

where $\zeta_{,}$ becomes a constant, and equal to $\alpha - \gamma$.

These equations are to be taken in connection with the equations of motion, and, as will presently be seen, will, along with these equations, assume the form of eight linear equations in a'', b'', $\xi_{,}, \eta_{,}$, ξ_1, η_1, ξ_2, η_2, with constant coefficients, reducible to four, by means of which the motion of the body will be completely determined, and the elements of its position assigned in finite and explicit functions of the time.

It would be easy to show, as Lagrange has done in the case of a body revolving and oscillating about a fixed point, that the centrifugal force of a body revolving on a surface nearly horizontal will throw its vertical axis to a finite distance from the fixed vertical, unless when either the rotation round the body's vertical is very small, in which case the distorsive moments of inertia F and G may be any whatever, or else when F and G are very small, and then the rotation round the vertical may be what we please. In both cases the form of the body and the distribution of its density may be such that the third distorsive moment of inertia H (which is brought into action only by the velocities p and q, and enters into the values of U, V and W, multiplied by these velocities only, or by their rates of increase) may be indefinitely great without affecting the truth of the solution.

Supposing then in the first place that r is very small, the values of p and q already found become $p = db''$, $q = -da''$, and the four equations last given (omitting hereafter the inferior accents of $\zeta_{,}$ and $\gamma_{,}$, as no longer wanted) are reduced to (37)

$$d\xi_1 = d\xi_, + \zeta dQ,$$
$$d\eta_1 = d\eta_, - \zeta dP;$$

$$d^2\xi_2 = d^2\xi_1,$$
$$d^2\eta_2 = d^2\eta_1:$$

whence we obtain

$$d^2\xi_2 = d^2\xi_, - \zeta da'' dt, \qquad d^2\eta_2 = d^2\eta_, + \zeta db'' dt.$$

By means of these expressions and the equation $\theta = g$, the two first equations of motion (34) become

$$\frac{d^2\xi_,}{dt^2} - \zeta\frac{da''}{dt} + \frac{g}{\lambda}\xi_, - ga'' = 0,$$

$$\frac{d^2\eta_,}{dt^2} - \zeta\frac{db''}{dt} + \frac{g}{\mu}\eta_, - gb'' = 0.$$

At the same time the equation $W = 0$ (15) becomes

$$Cd^2R + Fd^2a'' - Gd^2b'' = 0.$$

Substituting, in the expressions for U and V (15), db'' for p, $-da''$ for q, and for dr its value derived from the preceding equation, we shall find

$$(AC - G^2)\frac{d^2b''}{dt^2} + (CH + GF)\frac{d^2a''}{dt^2} - CMgB_{,,}\eta_, = 0,$$

$$(BC - F^2)\frac{d^2a''}{dt^2} + (CH + GF)\frac{d^2b''}{dt^2} - CMgA_{,,}\xi_, = 0;$$

which, together with the two equations above involving the same four variables, constitute four linear equations of the second order, with constant coefficients. It is well known that such equations are in all possible cases integrable in finite terms by the method of D'Alembert or other analogous pro-

cesses. (*Lacroix*, *Cal. Int.* Vol. II. p. 37.) In the course of this computation, into which the limits of the present communication will not allow me to enter, equations of limitation will arise showing the conditions of the oscillatory motions of the body. These equations will in general be expressed in the form of relations between the constants which determine the form and magnitude of the areolas of contact, the magnitude and density of the body and the position of its centre of gravity. Among the oscillatory motions possible, there is one of a peculiar nature which I do not recollect ever having seen remarked,—I mean when the motion is around a state of equilibrium, stable from the form of the moving body but unstable from the form of the supporting surface, or the contrary; as for example, when an ellipsoid is balanced on the outer surface of a sphere, the summit of the *shortest* axis of the ellipsoid being in contact with the highest point on the surface of the sphere. Into such a position we may conceive the ellipsoid to have descended from some assignable initial place of rest, or some combination of position and velocity. A motion would ensue which in a variety of cases would be oscillatory. The oscillations would however be liable to be broken by the application of the slightest force, and would be followed by the entire departure of the body from the place it occupied. These motions may be called *unstable oscillations*. They bear the same relation to *stable oscillations* that unstable does to stable equilibrium.

With respect to the four linear equations above obtained, I shall only add that in the present case they may be immediately reduced by eliminating $\xi_{,}$ and $\eta_{,}$ to two equations of the fourth order of the form

$$A\frac{d^4a''}{dt^4}+B\frac{d^4b''}{dt^4}+C\frac{d^2a''}{dt^2}+D\frac{d^2b''}{dt^2}+E\frac{da''}{dt}+Ga''=0,$$

$$A'\frac{d^4a''}{dt^4}+B'\frac{d^4b''}{dt^4}+C'\frac{d^2a''}{dt^2}+D'\frac{d^2b''}{dt^2}+F\frac{db''}{dt}+Hb''=0.$$

The eight arbitrary constants introduced by the integration of these equations are to be determined from the known values which the variables a'', b'', $\xi_{,}$, $\eta_{,}$ and their velocities are supposed to have at some given epoch of time. These eight arbitraries are not the only ones of which the body is susceptible. There will be ten in all, two being introduced by the equation $W = 0$, whose integral is

$$CR + Fa'' - Gb'' = \varepsilon t + \varepsilon',$$

the constants ε and ε' being functions of the values which a'', b'', R, and their velocities have at any given epoch.

Let us now suppose that the distorsive moments of inertia F and G are very small, in which case the rotation round the normal may be increased to any assignable rapidity without disturbing by that circumstance alone the smallness of the oscillatory excursions. The equation $W = 0$ will now be found reduced to $Cdr = 0$, whence $r =$ a constant quantity, and $R = rt + R'$, R' being the angular distance of the first body-axis from the first space-axis when $t = 0$. Equations (35) become at the same time

$$\begin{aligned} d\,\xi_1 &= d\,\xi_{,} - r\eta_{,} + \zeta(rb'' - da''), \\ d\,\eta_1 &= d\,\eta_{,} + r\xi_{,} - \zeta(ra'' + db''); \end{aligned}$$

$$\begin{aligned} d^2\xi_2 &= d^2\xi_1 - rd\eta_1, \\ d^2\eta_2 &= d^2\eta_1 + rd\xi_1, \end{aligned}$$

four linear equations which, in conjunction with the four equations of motion transformed by the substitution of the present values of p, q and r, will make up eight equations of the first degree (six being of the second and two of the first order) with constant coefficients. The equations may be completely integrated either by D'Alembert's method, by which we should be brought to twelve equations of the first order; or by eliminating the indefinite integrals ξ_1, η_1, ξ_2, η_2, and then proceeding by the method of exponential substitu-

tions. D'Alembert's method of integrating simultaneous linear equations is regarded by some of the first mathematicians of Europe as the best, and I have therefore introduced the equations (35); but if the direct substitution of exponential functions of the time be preferred (a method which has often the advantage of greater expedition), it would not be necessary to form these equations, as the values of $d^2\xi_2$, $d^2\eta_2$, $d^2\zeta_2$ are derivable from their equations of definition (10) in terms of the rotatory velocities and the coordinates, parallel to the body-axes, of the centre of gravity. For if we multiply by a, a', a'', the values of the second differentials of ξ, ξ', ξ'', the sum of the three products will be equal to $d^2\xi_2$ by the definition of this quantity, which is in fact the velocity which the point $O_{,}$ gains in every interval dt estimated in the direction which the body's first axis has at the *beginning* of that interval. It is because this acceleration is measured not on the variable axis itself, but on the *direction* which that axis had at the beginning of dt, that the sum of the elements $d^2\xi_2$ will not make up the velocity $d\xi_1$, nor the sum of the elements $d\xi_1$ the finite rate of increase of $\xi_{,}$. In consequence of these distinctions, many difficulties might arise in considering geometrically problems of the nature of the one before us; but they are always either avoided or explained by the adoption of analytical methods of solution, and I feel assured that the experience of those who are conversant with these methods will bear me out in saying that the necessity of even adverting to the difficulties of geometrical mechanics disappears precisely in proportion to the purity and generality of the analysis. While on this subject however I ought to remark that in consequence of this incompleteness of the values of $d\xi_1$, $d\eta_1$, $d\zeta_1$, and in the case of perfect rolling of $d\xi$, $d\xi'$, $d\xi''$, the application of Lagrange's Subsidiary Formula (*Méc. Anal.* Vol. I. p. 313) is inadmissible in such cases, and would lead to false results even if the velocities $d\xi$, $d\xi'$, $d\xi''$ be expressed in functions of the finite angles ψ, ϕ, θ and their velocities. In short his method is applicable only when the differential equations connecting the variables fulfil the conditions of integrability.

The values of the resolved partial accelerations of the centre of gravity found as above directed are

$$d^2\xi_2 = \begin{cases}(aa)d^2\xi_{,} + 2(ada)d\xi_{,} + (ad^2a)\xi_{,}\\ (ab)d^2\eta_{,} + 2(adb)d\eta_{,} + (ad^2b)\eta_{,}\\ (ac)d^2\zeta_{,} + 2(adc)d\zeta_{,} + (ad^2c)\zeta_{,},\end{cases}$$

$$d^2\eta_2 = \begin{cases}(ba)d^2\xi_{,} + 2(bda)d\xi_{,} + (bd^2a)\xi_{,}\\ (bb)d^2\eta_{,} + 2(bdb)d\eta_{,} + (bd^2b)\eta_{,}\\ (bc)d^2\zeta_{,} + 2(bdc)d\zeta_{,} + (bd^2c)\zeta_{,},\end{cases}$$

$$d^2\zeta_2 = \begin{cases}(ca)d^2\xi_{,} + 2(cda)d\xi_{,} + (cd^2a)\xi_{,}\\ (cb)d^2\eta_{,} + 2(cdb)d\eta_{,} + (cd^2b)\eta_{,}\\ (cc)d^2\zeta_{,} + 2(cdc)d\zeta_{,} + (cd^2c)\zeta_{,},\end{cases}$$

where the parentheses denote a sum of three quantities of which the first is included between the parentheses and the other two are similar and accented once and twice. These abridgments, combined with analogous ones for the sum of three quantities differing by a change of letters, might be used with great advantage in general inquiries into the phenomena of the progressive and rotatory motions of solid or fluid bodies; and I should have employed them throughout this paper, had I not been principally desirous of being clearly understood. In case several terms were to be included in the parentheses, an accent or inferior index might be annexed to the second parenthesis for the sake of obviating any ambiguity.

Substituting for the quantities in parentheses their values, all of which are given (6) and (7), we shall find

$$\begin{aligned} d^2\xi_2 &= d^2\xi_{,} - 2(rd\eta_{,} - qd\zeta_{,}) - \xi_{,}(q^2 + r^2) + \eta_{,}(pq - dr) + \zeta_{,}(rp + dq),\\ d^2\eta_2 &= d^2\eta_{,} - 2(pd\zeta_{,} - rd\xi_{,}) - \eta_{,}(r^2 + p^2) + \zeta_{,}(qr - dp) + \xi_{,}(pq + dr),\\ d^2\zeta_2 &= d^2\zeta_{,} - 2(qd\xi_{,} - pd\eta_{,}) - \zeta_{,}(p^2 + q^2) + \xi_{,}(rp - dq) + \eta_{,}(qr + dp).\end{aligned}$$

In the case of small oscillations, r at the same time being small, these become, as before

$$d^2\xi_2 = d^2\xi_{,} + \zeta\,dq, \qquad d^2\eta_2 = d^2\eta_{,} - \zeta\,dp.$$

If r is not small, it is constant, as we have seen, and we have

$$\begin{aligned} d^2\xi_2 &= d^2\xi_{,} - 2rd\eta_{,} - r^2\xi_{,} + \zeta(rp + dq), \\ d^2\eta_2 &= d^2\eta_{,} + 2rd\xi_{,} - r^2\eta_{,} + \zeta(rq - dp). \end{aligned}$$

Substituting for p and q their values (32) and employing the abridgments $\xi_{,} - \zeta a'' = u$, $\eta_{,} - \zeta b'' = v$, we shall find

$$\begin{aligned} d^2\xi_2 &= d^2u - 2rdv - r^2u, \\ d^2\eta_2 &= d^2v + 2rdu - r^2v. \end{aligned}$$

By means of these values, and the values of $\xi_{,}$ and $\eta_{,}$ obtained from the abridgments last employed, the two equations of progressive motion are converted into linear equations of the second order involving a'', b'', u, v and t. At the same time the two equations of rotatory motion are transformed, by the substitution of the values of p and q, into two other linear equations of the same order involving the same variables. In this way we shall obtain

$$\frac{d^2u}{dt^2} - 2r\frac{dv}{dt} + \lambda' u + \lambda'' g a'' = 0,$$

$$\frac{d^2v}{dt^2} + 2r\frac{du}{dt} + \mu' v + \mu'' g b'' = 0;$$

$$(\mathfrak{A}1)\frac{d^2a''}{dt^2} + (\mathfrak{A}2)\frac{d^2b''}{dt^2} + (\mathfrak{A}3)\frac{da''}{dt} + (\mathfrak{A}4)a'' + (\mathfrak{A}5)b'' + (\mathfrak{A}6)v + \mathfrak{F}r^2 = 0,$$

$$(B1)\frac{d^2b''}{dt^2} + (B2)\frac{d^2a''}{dt^2} + (B3)\frac{db''}{dt} + (B4)b'' + (B5)a'' + (B6)u + \mathfrak{G}r^2 = 0;$$

four linear equations with constant coefficients whose values are

$$\lambda' = r^2 - \frac{g}{\lambda}, \qquad \lambda'' = \frac{\zeta}{\lambda} - 1,$$

$$\mu' = r^2 - \frac{g}{\mu}; \qquad \mu'' = \frac{\zeta}{\mu} - 1;$$

$$\begin{array}{ll}
(A1) = \quad H, & (B1) = \quad H, \\
(A2) = \quad A, & (B2) = \quad B, \\
(A3) = \quad (A+B-C)r, & (B3) = -(A+B-C)r, \\
(A4) = \quad Hr^2, & (B4) = \quad Hr^2, \\
(A5) = \quad (C-B)r^2 - m'\zeta, & (B5) = \quad (C-A)r^2 - l'\zeta, \\
(A6) = -\ m' = -MgB_{,,}: & (B6) = -\ l' = -MgA_{,,}.
\end{array}$$

These equations may, by the elimination of u and v, be reduced to two of the fourth order, of eleven terms each, no term being wanting. They may be then completely integrated, and after the determination of the value of the ten arbitrary constants, eight of which are introduced by these equations and two others by the equation $W = 0$, the position of the body and all the phenomena of the motion will be expressed in terms of the sines and cosines of arcs proportional to the time. The conditions of oscillatory motion will also be expressed by equations of limitation arising during the process of determining the integrals.

I shall conclude this paper with an application of the preceding formulas to the determination of the small oscillatory motions of bodies of any figure, law of density, and areola of contact, rolling with the three rotations on a surface which from some slight asperity or other cause prevents entirely and in all directions the sliding motion of the body, while in other respects it leaves it free to rock, pitch and spin, with any combination of these motions consistent with a small declination of the natural vertical of the body from the vertical of equilibrium. I ought to remark that this motion, although more resembling the actual oscillations of supported bodies, differs from them materially in the circumstance that the friction is supposed not to interfere with the motion round the normal, whereas this cause undoubtedly cooperates with the resisting medium to retard the horizontal rotation of the

body until it ceases altogether. What I am about to offer therefore must be considered, like every advance which has hitherto been made, as merely a step towards the determination of the actual phenomena. It would not be difficult to include in the next place the moments of the forces which resist the rotation round the normal, but this must form the subject of another dissertation.

The fundamental equations of condition resulting from the definition of the species of motion we are now considering are, as we have seen (25)

$$\begin{aligned} 0 &= \delta\xi_1 - y_{,}\delta R + z_{,}\delta Q, \\ 0 &= \delta\eta_1 - z_{,}\delta P + x_{,}\delta R, \\ 0 &= \delta\zeta_1 - x_{,}\delta Q + y_{,}\delta P. \end{aligned}$$

Substituting these values of the variations of the position of $O_{,}$ the centre of gravity in the general dynamical equation, there will result an equation of the form

$$(P)\,\delta P + (Q)\,\delta Q + (R)\,\delta R = 0;$$

in which the variations are now arbitrary, giving us therefore three equations of motion to be taken in conjunction with the three above, namely,

$$(P) = 0, \qquad (Q) = 0, \qquad (R) = 0:$$

or, writing out these equations at full length, in the case of common gravity,

$$U + M\left(\frac{d^2\eta_2}{dt^2} - gb''\right)z_{,} - M\left(\frac{d^2\zeta_2}{dt^2} - gc''\right)y_{,} = 0,$$

$$V + M\left(\frac{d^2\zeta_2}{dt^2} - gc''\right)x_{,} - M\left(\frac{d^2\xi_2}{dt^2} - ga''\right)z_{,} = 0,$$

$$W + M\left(\frac{d^2\xi_2}{dt^2} - ga''\right)y_{,} - M\left(\frac{d^2\eta_2}{dt^2} - gb''\right)x_{,} = 0;$$

expressions which are true whether friction be considered or not, and independently of all hypotheses of friction.

If the body remains always nearly upright, these become

$$U + M\gamma\left(\frac{d^2\eta_2}{dt^2} - gb''\right) + gy_, = 0,$$

$$V - M\gamma\left(\frac{d^2\xi_2}{dt^2} - ga''\right) - gx_, = 0,$$

$$W = 0.$$

These equations furnish the same relations between F, G and r as those obtained before. Either the rotation round the natural vertical, or else those moments of inertia which would (when made effective by a swift rotation) displace that vertical, must be very small. If r is very small, the equations of condition of perfect rolling are reduced to

$$d\xi_1 = -\gamma q dt, \qquad d\eta_1 = \gamma p dt.$$

Substituting these values in equations (37), and recollecting that $\zeta + \gamma = \alpha''$, we shall find

$$d\xi_, = -\alpha'' q dt, \qquad d\eta_, = \alpha'' p dt.$$

But when r is small we have

$$d^2\xi_2 = d^2\xi_, + \zeta dq dt, \qquad d^2\eta_2 = d^2\eta_, - \zeta dp dt.$$

Therefore

$$d^2\xi_2 = -\gamma dq dt, \qquad d^2\eta_2 = \gamma dp dt;$$

equations which are verified by the equation formerly obtained (37) when r is small, $d^2\xi_2 = d^2\xi_1$, $d^2\eta_2 = d^2\eta_1$. Finally, these last equations become, in consequence of the values which p and q acquire when r is small,

$$d^2\xi_2 = \gamma d^2 a'', \qquad d^2\eta_2 = \gamma d^2 b'';$$

by which means the equations of motion are reduced to equations with constant coefficients, namely,

$$U + M\gamma^2 \frac{d^2 b''}{dt^2} - Mg\gamma b'' + Mgy_, = 0,$$

$$V - M\gamma^2 \frac{d^2 a''}{dt^2} + Mg\gamma b'' - Mgx_, = 0.$$

Where the oscillations take place upon a spherical areola of support, which will include oscillations on a horizontal plane, we have $x_, = l\xi_{,}$, $y_, = m\eta_{,}$, and therefore, by preceding formulas,

$$dx_, = l\alpha da'', \qquad dy_, = m\alpha db''.$$

Integrating, and denoting by χ and ψ the arbitrary constants, there results

$$x_, = l\alpha a'' + \chi, \qquad y_, = m\alpha b'' + \psi;$$

which being substituted in the above equations of motion give two equations of the second order in a'', b'' and t of the form

$$(A1)\frac{d^2a''}{dt^2} + (A2)\frac{d^2b''}{dt^2} + (A3)a'' + (A4) = 0,$$

$$(B1)\frac{d^2b''}{dt^2} + (B2)\frac{d^2a''}{dt^2} + (B3)b'' + (B4) = 0,$$

where the coefficients may be readily determined, as U and V have now the same value as before when there was no friction and when the rotation round the normal was at the same time small. These coefficients being constant, the equations may be completely integrated in finite terms, four arbitrary constants being introduced by the integration, which together with χ and ψ introduced by the last integrals obtained, and ε and ε' arising as we have already seen from the integration

of $W = o$, make up eight in all, being two less than when the body was not restricted to the peculiar motion to which we now suppose it to be subject.

Lastly, let F and G be very small. The equation $W = o$ will now give us, as if there were no friction, $r =$ any arbitrary constant, and $R = rt + R'$. At the same time we have

$$p = ra'' + db'', \quad d\xi_1 = d\xi_, - r\eta_, + \zeta q, \quad d^2\xi_2 = d^2\xi_1 - rd\eta_1,$$
$$q = rb'' - da'', \quad d\eta_1 = d\eta_, + r\xi_, - \zeta p, \quad d^2\eta_2 = d^2\eta_1 + rd\xi_1,$$

and, by the equations of perfect rolling,

$$d\xi_1 = ry_, - \gamma q,$$
$$d\eta_1 = -rx_, + \gamma p;$$

whence

$$d^2\xi_2 = r^2x_, + rdy_, + \gamma(d^2a'' + 2rdb'' - r^2a''),$$
$$d^2\eta_2 = r^2y_, - rdx_, + \gamma(d^2b'' - 2rda'' - r^2b'').$$

By comparing the two values above given for each of the quantities $d\xi_1$ and $d\eta_1$, we obtain, after replacing $\zeta + \gamma$ by α,

$$r(x_, + \xi_,) = \alpha p - d\eta_,,$$
$$r(y_, + \eta_,) = \alpha q + d\xi_,.$$

When the supporting areola is spherical, these become

$$r\lambda x_, = \alpha m(ra'' + db'') - dy_,,$$
$$r\mu y_, = \alpha l(rb'' - da'') + dx_,,$$

where λ and μ are abridgments for $l + \frac{l}{m}$ and $m + \frac{m}{l}$.

By means of the preceding values of $d^2\xi_2$ and $d^2\eta_2$ it will be seen that the first and second equations of motion are transformed into equations of the second order involving a'', b'', $x_,$, y and t with constant coefficients. These, in con-

junction with the two last equations which are of the first order involving the same variables, will enable us to determine fully and by finite integrations all the circumstances of the oscillatory motion. The arbitrary constants will be eight in number, six of them being brought in by the four equations just referred to, and two of them, R' and r, arising from the third equation of motion $W = 0$. These four equations will be found to be

$$\mathcal{A}_1 \frac{d^2a''}{dt^2} + \mathcal{A}_2 \frac{d^2b''}{dt^2} + \mathcal{A}_3 \frac{da''}{dt} + \mathcal{A}_4 \frac{dx_{\prime}}{dt} + \mathcal{A}_5 a'' + \mathcal{A}_6 b'' + \mathcal{A}_7 y_{\prime} + \mathcal{A}_8 = 0,$$

$$B_1 \frac{d^2b''}{dt^2} + B_2 \frac{d^2a''}{dt^2} + B_3 \frac{db''}{dt} + B_4 \frac{dy_{\prime}}{dt} + B_5 b'' + B_6 a'' + B_7 x_{\prime} + B_8 = 0,$$

$$C_1 \frac{da''}{dt} + C_2 \frac{dx_{\prime}}{dt} + C_3 b'' + C_4 y_{\prime} = 0,$$

$$D_1 \frac{db''}{dt} + D_2 \frac{dy_{\prime}}{dt} + D_3 a'' + D_4 x_{\prime} = 0;$$

the value of their coefficients being as follows,

$$\begin{aligned}
\mathcal{A}_1 &= (\mathcal{A}1), & B_1 &= (B1),\\
\mathcal{A}_2 &= (\mathcal{A}2) + M\gamma^2, & B_2 &= (B2) + M\gamma^2,\\
\mathcal{A}_3 &= (\mathcal{A}3) - 2M\gamma^2 r, & B_3 &= (B3) + 2M\gamma^2 r,\\
\mathcal{A}_4 &= - M\gamma r, & B_4 &= M\gamma r,\\
\mathcal{A}_5 &= (\mathcal{A}4), & B_5 &= (B4),\\
\mathcal{A}_6 &= (\mathcal{A}5) - M\gamma^2 r^2 - Mg\gamma, & B_6 &= (B5) - M\gamma^2 r^2 - Mg\gamma,\\
\mathcal{A}_7 &= M\gamma r^2 + Mg, & B_7 &= M\gamma r^2 + Mg.\\
\mathcal{A}_8 &= Fr^2; & B_8 &= Gr^2;
\end{aligned}$$

$$\begin{aligned}
C_1 &= \alpha l, & D_1 &= \alpha m,\\
C_2 &= -1, & D_2 &= -1,\\
C_3 &= -\alpha l r, & D_3 &= \alpha m r,\\
C_4 &= r\mu; & D_4 &= -r\lambda.
\end{aligned}$$

The principles and formulas detailed in this memoir will also enable us to determine completely all the circumstances of the motion of any solid of revolution rolling or spinning with or without friction upon a horizontal plane, its axis being supposed to form at all times a very small but variable angle with the plane. The length to which this paper has extended itself obliges me, however, to defer for the present the further consideration of this subject. I shall confine myself therefore to the remark, that in some of these cases, and in a variety of others, the equations given at the foot of page 377 may be presented with advantage in the following form:—

$$\frac{\mathrm{d}(a\,P'+b\,Q'+c\,R')}{M\mathrm{d}t} = (x'-\xi')\frac{\mathrm{d}^2\xi''}{\mathrm{d}t^2}-(x''-\xi'')\frac{\mathrm{d}^2\xi'}{\mathrm{d}t^2}-g(x'-\xi'),$$

$$\frac{\mathrm{d}(a'P'+b'Q'+c'R')}{M\mathrm{d}t} = (x''-\xi'')\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}-(x-\xi)\frac{\mathrm{d}^2\xi''}{\mathrm{d}t^2}+g(x-\xi),$$

$$\frac{\mathrm{d}(a''P'+b''Q'+c''R')}{M\mathrm{d}t} = (x-\xi)\frac{\mathrm{d}^2\xi'}{\mathrm{d}t^2}-(x'-\xi')\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}.$$

ERRATA.

Page 342, *line* 19, *for* "(6) and (10)" *read* "(6), (7) and (10)".

.. 352, .. 15, *for* "$\alpha\delta Q$" *read* "$\alpha\delta R$".

.. .. .. 16, *for* "$\beta\delta Q$" *read* "$\beta\delta P$".

.. 354, .. 6 *to* 11. *These formulas should be numbered* "(27)".